Reelle Analysis - Serie 1

Aufgabe 1: (i) We first check that $\sigma_1 = \frac{(-1)^0 + (-1)^1}{1} = 0$

$\sigma_2 = \frac{(-1)^0 + (-1)^1 + (-1)^2}{2} = \frac{1}{2} \ldots$

from which we infer by induction that $\sigma_n = \begin{cases} 0, & n \text{ odd} \\ \frac{1}{n}, & n \text{ even (gerade)} \end{cases}$

It follows that $|\sigma_n| \leq \frac{1}{n} \longrightarrow 0$, as $n \to +\infty$.

(ii) Suppose that (s_n) converges and $s := \lim_{n \to +\infty} s_n \in \mathbb{R}$. $(*)$

We prove that $\lim_{n \to +\infty} \sigma_n = s$.

Let $\varepsilon > 0$ be fixed. By definition from $(*)$: $\exists n_0 \in \mathbb{N}, \forall n \geq n_0 : |s_n - s| < \varepsilon$ $(**)$

Then: $\sigma_n - s = \sum_{k=0}^{n-1} \frac{s_k}{n} - s = \frac{1}{n} \sum_{k=0}^{n-1} (s_k - s)$. By the triangle inequality,

for $n \in \mathbb{N}^*$

for $n > n_0$: $|\sigma_n - s| \leq \frac{1}{n} \underbrace{\sum_{k=0}^{n_0 - 1} |s_k - s|}_{=: C(\varepsilon) > 0} + \frac{1}{n} \sum_{k=n_0}^{n-1} \underbrace{|s_k - s|}_{< \varepsilon \text{ from } (**)}$

constant tha depends only on n_0, hence on ε.

so: $|\sigma_n - s| \leq \underbrace{\frac{C(\varepsilon)}{n}}_{\to 0} + \underbrace{\frac{n - n_0}{n}}_{\leq 1} \varepsilon < 2\varepsilon$, $\forall n \geq n_2 := \max(n_0, n_1)$.

$\longrightarrow 0$ as $n \to +\infty$

hence $< \varepsilon$ for $n \geq n_1$ where $n_1 \in \mathbb{N}$

By definition, this proves that $s = \lim_{n \to +\infty} \sigma_n$.

(iii) False. Take for example $s_n = \begin{cases} -\sqrt{n-1}, & n \text{ odd} \\ \sqrt{n}, & n \text{ even} \end{cases}$, $n \in \mathbb{N}$.

clearly $|s_n| \to +\infty$ as $n \to +\infty$ so (s_n) is not bounded, but by induction: $\sigma_n = \begin{cases} 0, & n \text{ odd} \\ \frac{1}{\sqrt{n}}, & n \text{ even} \end{cases}$ so that $|\sigma_n| \leq \frac{1}{\sqrt{n}} \longrightarrow 0$ as $n \to +\infty$.

and (s_n) is thus Cesàro - summable

Aufgabe 2:

Given $f, g \in R(\mathbb{T})$, $f*g \in R(\mathbb{T})$.

•) $$f*g(x+2\pi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\underbrace{g(x+2\pi-t)}_{=g(x-t) \text{ since } g \text{ is } 2\pi\text{-periodic}}\,dt = f*g(x),$$ so $f*g$ is 2π-periodic.

•) Riemann's integrability follows from Fubini's theorem:

$$\int_{\mathbb{T}} |(f*g)(x)|\,dx = \frac{1}{2\pi}\int_{\mathbb{T}} \left|\int_{-\pi}^{\pi} f(t)g(x-t)\,dt\right| dx \le \frac{1}{2\pi}\int_{\mathbb{T}}\int_{-\pi}^{\pi} |f(t)|\,|g(x-t)|\,dt\,dx$$

$$\underset{\text{Fubini}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)| \Big(\underbrace{\int_{\mathbb{T}} |g(x-t)|\,dx}_{=\int_{\mathbb{T}}|g|}\Big)\,dt \le \frac{1}{2\pi}\int_{\mathbb{R}} |f| \int_{-\pi}^{\pi} |g| < +\infty$$

translation invariant by 2π-periodicity: $\int_{\mathbb{T}} = \int_{-\pi}^{\pi}$

(i) •) $$f*g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(x-t)\,dt \underset{\uparrow}{=} \frac{1}{2\pi}\int_{x+\pi}^{x-\pi} f(x-s)g(s)\,(-ds)$$

Substitution: $s = x-t$, $ds = -dt$

$$= \frac{1}{2\pi}\int_{x-\pi}^{x+\pi} f(x-s)g(s)\,ds \underset{\substack{\text{translation invariance} \\ \text{by } 2\pi\text{-periodicity}}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-s)g(s)\,ds = g*f(x).$$

(ii) $$\widehat{f*g}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f*g(t)\,e^{-ikt}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)g(t-s)\,ds\right)\underbrace{e^{-ikt}}_{=e^{-ik(t-s)}\cdot e^{-iks}}\,dt$$

$$\underset{\text{Fubini}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)e^{-iks}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(t-s)e^{-ik(t-s)}\,dt\right) ds \underset{\substack{\uparrow \\ \text{Substitution}}}{=} \hat{f}(k)\,\hat{g}(k)$$

Aufgabe 3:

We suppose that $f_n \longrightarrow f$ uniformly on $[-\pi,\pi]$

i.e. $\forall \varepsilon>0,\ \exists n_0 \in \mathbb{N},\ \forall n \ge n_0:$ $$\underbrace{\|f_n - f\|_{L^\infty}}_{=: \sup_{t\in[-\pi,\pi)} |f_n(t)-f(t)|} < \varepsilon \qquad (*)$$

It readily implies:

$$\|f_n - f\|_{L^2(-\pi,\pi)} \le \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\sup_{x\in[-\pi,\pi]} |f_n(x)-f(x)|^2}_{<\varepsilon^2 \text{ for } n \ge n_0}\,dt\right)^{1/2} \le \varepsilon, \quad \forall n \ge n_0.$$

hence $f_n \longrightarrow f$ in L^2.

Aufgabe α: Let $a>0$ and $f:\mathbb{R}\to\mathbb{R}$ be the 2π-periodic extension of $f(x)=\cosh(ax)$, $x\in[-\pi,\pi]$

Recall: $\cosh(x)=\dfrac{e^x+e^{-x}}{2}$ $(=\mathrm{ch}(x))$. We compute for $k\in\mathbb{Z}$:

$$\hat f(k)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}\left(e^{ax}+e^{-ax}\right)e^{-ikx}\,dx=\frac{1}{4\pi}\int_{-\pi}^{\pi}\left[e^{x(a-ik)}+e^{-x(a+ik)}\right]dx$$

$$=\frac{1}{4\pi}\left\{\frac{1}{a-ik}e^{x(a-ik)}\Big|_{-\pi}^{\pi}+\frac{1}{-(a+ik)}e^{-x(a+ik)}\Big|_{-\pi}^{\pi}\right\}\quad\left(\text{note that } e^{\pm ik\pi}=(-1)^k\right)$$

$$=\frac{1}{4\pi}\left\{\frac{(-1)^k}{a-ik}\left(e^{a\pi}-e^{-a\pi}\right)+\frac{(-1)^k}{-(a+ik)}\left(e^{-a\pi}-e^{a\pi}\right)\right\}$$

$$=\frac{(-1)^k}{4\pi}\left(e^{a\pi}-e^{-a\pi}\right)\cdot\frac{(a+ik)+(a-ik)}{|a+ik|^2}=\frac{(-1)^k}{2\pi}\left(e^{a\pi}-e^{-a\pi}\right)\frac{a}{a^2+k^2}$$

$$=\frac{(-1)^k}{\pi}\sinh(a\pi)\frac{a}{a^2+k^2}\qquad\text{where } \sinh(x)=\frac{e^x-e^{-x}}{2}.$$

By Dirichlet theorem: $f\in\mathcal{R}(\mathbb{T})$ and continuous $\Rightarrow$ $f(x)=\dfrac{\sinh(\pi a)}{\pi}\displaystyle\sum_{k\in\mathbb{Z}}{}'\frac{(-1)^k a}{a^2+k^2}e^{ikx}$ for $x\in[-\pi,\pi]$.

In particular:

$$\cosh(\pi a)=f(\pi)=\frac{\sinh(\pi a)}{\pi}\sum_{k\in\mathbb{Z}}{}'\frac{(-1)^k a}{a^2+k^2}\cdot(-1)^k=\frac{\sinh(\pi a)}{\pi}\left\{\frac{1}{a}+2\sum_{k=1}^{\infty}{}'\frac{a}{a^2+k^2}\right\}$$

whence: $$\sum_{k=1}^{\infty}{}'\frac{1}{a^2+k^2}=\frac{\pi}{2a}\coth(\pi a)-\frac{1}{2a^2}\qquad\text{where } \coth(x)=\frac{\sinh(x)}{\cosh(x)}.$$