Reelle Analysis - Serie 4

A13:

(i) $\mu_1(A) = \# A = $ nb of elements in $A \in [0,\infty) \cup \{\infty\}$.

Let $A, B \subset X$. If $\# B < +\infty$: $B = (B\cap A) \cup (B\setminus A)$ is the decomposition of B in two disjoint subsets, in which case counting elements in B we find:

$$\mu_1(B) = \# B = \#(B\cap A) + \#(B\setminus A) = \mu_1(B\cap A) + \mu_1(B\setminus A). \quad (*)$$

If $\# B = +\infty$ then either $\#(B\cap A) = +\infty$ or $\#(B\setminus A) = +\infty$ so that in any case $(*)$ holds by the convention that: $\infty = \infty + x, \ \forall x \in [0,\infty)\cup\{\infty\}$.

(ii) $A \subset X$ is μ_2-measurable $\Leftrightarrow A = \emptyset$ or $A = X$

"$\Leftarrow$" If $A = \emptyset$ (or X), then $A^c = X$ (or $\emptyset$), hence $B\cap A = \emptyset$ (or B) while $B\setminus A = B$ (or $\emptyset$)
In both cases the identity $\mu_2(B\cap A) + \mu_2(B\setminus A) = \mu_2(B)$ holds $\forall B \subset X$.

"$\Rightarrow$" Take $B = X$ in the def. of μ_2-measurability for A:

$$1 = \mu_2(X) = \mu_2(X\cap A) + \mu_2(X\setminus A) = \mu_2(A) + \mu_2(A^c)$$

For $X \neq \{x\}$ and $A \neq \emptyset$ and $A \neq X$ we thus find $1 = 2$. A contradiction

A14: (i) λ is a pre-measure (Prämaß)

(1) $\lambda(\emptyset) = 0$ by def.

(2) σ-additivity: A pairwise disjoint collection $(A_k)_{k\in\mathbb{N}} \subset \mathcal{A} = \{\emptyset; X\}$ can only be either one of the following: $\{\emptyset\}$; $\{X\}$; $\{\emptyset; X\}$ or $\{X; \emptyset\}$ ("subsets" of a set whose 2 elements are $\emptyset$ and X). One readily checks these 4 decompositions satisfy to σ-additivity: $A_1 = \emptyset$: $\lambda(\emptyset) = \lambda(\emptyset)$; $A_1 = \emptyset, A_2 = X$: $\lambda(X) = \lambda(\emptyset) + \lambda(X)$
$A_2 = X$: $\lambda(X) = \lambda(X)$; $A_1 = X; A_2 = \emptyset$: $\lambda(X) = \lambda(X) + \lambda(\emptyset)$.

(ii) λ is σ-finite iff $X = \bigcup_{k\in\mathbb{N}} A_k$ with $\lambda(A_k) < +\infty$.

Here $X = A_1$ with $\lambda(A_1) = 1 < +\infty$. CCL: $\boxed{\lambda \text{ is a } \sigma\text{-finite pre-measure}}$

(iii) The possible outputs for $\bigcup_{j=1}^{\infty} A_j$, $A_j \in \{\emptyset; X\}$ are $\emptyset$ or X.

So: $\mu(A) = \inf\left\{ \sum_{j=1}^{\infty} \lambda(A_j) : A \subset \bigcup_{j=1}^{\infty} A_j,\ A_j = \emptyset \text{ or } X \right\}$.

Suppose $A \neq \emptyset$. then $\emptyset \neq A \subset \bigcup_{j=1}^{\infty} A_j$, $A_j = \emptyset$ or $X \Longrightarrow \exists j_0 \geq 1$ s.t. $A_{j_0} = X$.

which implies: $\sum_{j=1}^{\infty} \lambda(A_j) \geq \lambda(A_{j_0}) = 1$. Taking the infimum over all possible such collections $(A_j)_{j\geq 1}$ we find $\mu(A) \geq 1$

Moreover, using the decomposition: $A_1 = X$, $A_k = \phi$, $\forall k \geq 2$, we see by def. of μ that $\underline{\mu(A) \leq 1}$ so we conclude that $\mu(A) = 1$.

We easily see that $\underline{\mu(\phi) = 0}$. We have: $\underline{\text{ccl}}$: $\boxed{\mu(A) = \begin{cases} 1, & A \neq \phi \\ 0, & A = \phi \end{cases} = \mu_2(A)}$

(iv) From $\underline{\text{A13 (ii)}}$ the σ-algebra of $\mu = \mu_2$-measurable sets is $\Sigma = \mathcal{A} = \{\phi, X\}$.

(v) λ is a pre-measure such that its $\underline{\text{induced measure}}$ μ satisfies:

$$\mathcal{L}^1_{|\mathcal{A}} = \lambda_{|\mathcal{A}} \underset{\substack{\text{Carathéodory-Hahn} \\ \text{Th.}}}{\Longrightarrow} \mathcal{L}^1_{|\Sigma} = \mu_{|\Sigma}$$

but $\mathcal{L}^1([0,\frac{1}{2}]) = \frac{1}{2} \neq 1 = \mu([0,\frac{1}{2}])$ is no - contradiction since $[0,\frac{1}{2}] \notin \Sigma = \{\phi, [0,1]\}$

$\underline{\text{A15:}}$ $\underline{\text{(i) } \tilde{\mathcal{B}} \text{ is a } \sigma\text{-algebra}}$

$\underline{\text{Def:}}$ $\mathcal{A}$ $\underline{\sigma\text{-algebra}}$ if:
(1) $X \in \mathcal{A}$ (2) $A \in \mathcal{A} \Rightarrow X \setminus A \in \mathcal{A}$
(3) $A_k \in \mathcal{A}$, $k \in \mathbb{N} \Rightarrow \bigcup_{k\in\mathbb{N}} A_k \in \mathcal{A}$

(1) $X = \underset{\underset{(\{x\in X:\, f(x)\in Y\})}{\parallel}}{f^{-1}(Y)} \in f^{-1}(\mathcal{B})$ since $Y \in \mathcal{B}$ by (1) for $\mathcal{B}$.

(2) If $A = f^{-1}(B) \in f^{-1}(\mathcal{B})$ then $X \setminus A = \{x \in X: \underset{\underline{\text{i.e.}}\ f(x)\in Y\setminus B}{f(x) \notin B}\} = f^{-1}(Y \setminus B) \in f^{-1}(\mathcal{B})$

since $B \in \mathcal{B} \Rightarrow Y \setminus B \in \mathcal{B}$.

(3) Let $A_k = f^{-1}(B_k) \in f^{-1}(\mathcal{B})$, $k \in \mathbb{N}$. Then:

$$\bigcup_{k\in\mathbb{N}} A_k = \bigcup_{k\in\mathbb{N}} \{x\in X: f(x) \in B_k\} = \{x \in X: f(x) \in \bigcup_{k\in\mathbb{N}} B_k\} = f^{-1}\Big(\bigcup_{k\in\mathbb{N}} B_k\Big) \in f^{-1}(\mathcal{B})$$

since $B_k \in \mathcal{B} \Rightarrow \bigcup_{k\in\mathbb{N}} B_k \in \mathcal{B}$ by (3).

(ii) $\underline{\tilde{\mathcal{A}} \text{ is not always a } \sigma\text{-algebra}}$ Counter-example: $X = Y = \mathbb{R}$ and $\mathcal{A} = \mathcal{B} =$ Borel algebra. Then $f(x) = x^2$ implies that:

$\mathbb{R} \in \mathcal{A}$ but $\mathbb{R} \notin f(\mathcal{A}) \subset \{A \subset [0, +\infty[\}$.

(Or $\mathbb{R} \setminus f(\mathbb{R}) =]-\infty, 0[\notin \{f(A): A \subset \mathbb{R}\}$.)

$\underline{\text{A16}}$ $\mathcal{A} = \{A \subset \mathbb{N}: A \text{ or } A^c \text{ is infinite}\}$.

$\underline{\text{(i) } \mathcal{A} \text{ is an algebra:}}$ in the def. of a σ-algebra, we replace (3) by (3)':

(1) $\mathbb{N} \in \mathcal{A}$ since $\mathbb{N}^c = \phi$ has 0 element so is finite.

(2) If $A \in \mathcal{A}$ then A or A^c is finite $\underline{\text{i.e.}}$ $(A^c)^c$ or A^c is finite, so $A^c \in \mathcal{A}$.

(3)' If $A_k \in \mathcal{A}$, $1 \leq k \leq N$.

$\underline{\text{Either } \forall 1 \leq k \leq N,\ \# A_k < +\infty:}$ and so $\# \bigcup_{k=1}^{N} A_k < +\infty$ so $\bigcup_{k=1}^{N} A_k \in \mathcal{A}$

(a finite union of finite sets is finite)

Or there is at least one $1 \leq k_0 \leq N$ s.t. $\# A_{k_0} = +\infty$.

Then let $I := \{ 1 \leq k \leq N \quad \# A_k = +\infty \}$. Up to renaming $A_1 := \bigcup_{k \in I} A_k$,

we may assume that $\# A_1 = +\infty$ and $\# A_k < +\infty$, $\forall\, 2 \leq k \leq N$.

From $\left(\bigcup_{k=1}^{N} A_k \right)^c = \bigcap_{k=1}^{N} A_k^c \subset A_1^c$ with $\# A_1^c < +\infty$ we deduce $\bigcup_{k=1}^{N} A_k \in \mathcal{A}$.

(ii) Condition (3) fails

Take for instance $A_k = \{ 2k \}$, $k \in \mathbb{N}$. Then:

$$A := \bigcup_{k \in \mathbb{N}} A_k = 2\mathbb{N} \notin \mathcal{A}$$ since $A^c = 2\mathbb{N} + 1$ and neither A nor A is finite.

(iii) By definition; if λ were a pre-measure:

$$\lambda(\mathbb{N}) = 2 - \sum_{k \in \mathbb{N}^c}' \frac{1}{k^2} = 2$$

$$\shortparallel$$

$$\lambda\left(\bigcup_{k \in \mathbb{N}} \{k\} \right) \underset{(2)}{=} \sum_{k \in \mathbb{N}}' \lambda(\{k\}) = \sum_{k \in \mathbb{N}}' \frac{1}{k^2} = \frac{\pi^2}{6}.$$ A contradiction

λ pre-measure