A GAME-THEORETIC PROOF OF CONVEXITY PRESERVING PROPERTIES FOR MOTION BY CURVATURE

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Abstract. In this paper, we revisit the convexity preserving properties for the level set mean curvature flow equation by using the game-theoretic approximation established by Kohn and Serfaty (2006). Our new proofs are based on investigating game strategies or iterated applications of dynamic programming principle, without invoking deep partial differential equation theory. We also use this method to study convexity preserving for the Neumann boundary problem.

1. Introduction

We are interested in a game interpretation of convexity preserving properties for the level set curvature flow equation in the plane:

$$\text{(MCF)} \quad \begin{cases} u_t - |\nabla u| \operatorname{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) = 0 & \text{in } \mathbb{R}^2 \times (0,\infty), \quad (1.1) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^2, \quad (1.2) \end{cases}$$

where u_0 is assumed to be a uniformly continuous function in $\mathbb{R}^2$. The existence, uniqueness and stability of viscosity solutions to (MCF) are studied in [14, 10]; see also [16].

The convexity preserving property for mean curvature flow was originally discovered by Huisken [22], where it was shown that an evolving surface by mean curvature stays convex if the initial surface is convex; see also [15] for the two dimensional case in detail. The convexity of a surface here means that the surface is the boundary of a convex body.

This property was later formulated in terms of level set method in [14] and [17]. They proved the following result.

Theorem 1.1. *Suppose u_0 is Lipschitz continuous function on $\mathbb{R}^n$. Let u be the unique viscosity solution of (MCF). If the set $\{x \in \mathbb{R}^n : u_0(x) \geq 0\}$ is convex, then the set $\{x \in \mathbb{R}^n : u(x,t) \geq 0\}$ is also convex for any $t \geq 0$.*

Their result is based on a regularization of the mean curvature operator for the corresponding stationary problem in a convex domain $\Omega \subset \mathbb{R}^n$, that is,

$$\text{(SP)} \quad \begin{cases} -\Delta_1^G u = -|\nabla u| \operatorname{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) = 1 & \text{in } \Omega, \quad (1.3) \\ u = 0 & \text{on } \partial\Omega, \quad (1.4) \end{cases}$$

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and an application of more classical convexity maximum principle by Korevaar [30] and Kennington [27] for C^2 solutions of uniformly elliptic equations. In two dimensions, Barron, Goebel and Jensen [8] recently gave a PDE proof for the convexity of level sets for any strict subsolution or strict supersolution of $-\Delta_1^G u = 0$.

The convexity (concavity) preserving property of viscosity solutions was studied for a more general class of degenerate parabolic equations in [17], where the authors proved a general result including the following special case for mean curvature flow.

Theorem 1.2. *Suppose u_0 is a concave and Lipschitz continuous function on $\mathbb{R}^n$. Let u be the unique viscosity solution of (MCF). Then $u(x,t)$ is also concave in x for any $t \geq 0$.*

The argument relies on a generalized version of convexity comparison principle; we refer to [24] for related results. More precisely, it is shown, by adding more variables in the usual comparison theorem, that the continuous solution u satisfies

$$u(x,t) + u(y,t) - 2u(z,t) \leq |x + y - 2z|$$

for any $x, y, z \in \mathbb{R}^n$ and $t \geq 0$ if the same holds initially:

$$u_0(x) + u_0(y) - 2u_0(z) \leq |x + y - 2z|.$$

The concavity preserving property is then a special case when $z = (x+y)/2$. We remark that the convexity of solutions to various PDEs has also been extensively studied in [25, 26, 40, 12, 2] etc.

In this work, we give new proofs of Theorem 1.1 and Theorem 1.2 for the case $n = 2$, applying few PDE methods but instead using game-theoretic approximations provided by Kohn and Serfaty [28]. Our proofs are therefore very different from the works mentioned above. It is known [28] that there exists a family of games, whose value functions u^ε converge, as $\varepsilon \to 0$, to the solution u of (MCF) under the assumption (A). Thanks to the game interpretation, we may look to the convexity preserving property of u^ε rather than that of u. For the convenience in connecting with the existing game interpretation, we assume hereafter that

(A) u_0 is a bounded and uniformly continuous function in $\mathbb{R}^2$ satisfying that $u_0 \geq a$ and $u_0 - a$ has compact support K_a for some constant $a \in \mathbb{R}$.

We first discuss the convexity preserving property for level sets as discussed in Theorem 1.1. We show that any superlevel set $u^\varepsilon(\cdot, t)$ is approximately convex up to a small error in terms of ε if the level set of u_0 is convex. The idea is as follows. We first notice, for example, that the super zero superlevel set of $u^\varepsilon(x,t)$ in the game can be viewed as the set of all initial positions from which the game position can never depart from the zero superlevel set of u_0 in time t. Therefore, it suffices to show that, if the game position cannot exit the super c-level set of u_0 in time t starting either from x or from y, it cannot do it from $(x+y)/2$ either.

This follows easily from the game structure in two dimensions. The repeated rule of the game is particularly simple in $\mathbb{R}^2$. It consists of a first choice of a unit vector v by Player I and the other choice of a sign $b = \pm 1$ by Player II. The game position then changes by $\sqrt{2}\varepsilon b v$, depending on the choices. Under this rule, one can easily find that in the game from $(x+y)/2$ with duration t, Player I can always choose to stay on the line segment connecting x and y until it is forced to approach either of them. Since such a process costs time, the remaining time is then not enough for the game position to leave the region from either x or y. This leads us to the conclusion. Note that in this case we do not assume the convexity/concavity of u_0 itself but only the convexity of its superlevel sets. Our game-theoretic argument can be applied to understand the convexity of level sets of the solution to (SP) as well, but, as emphasized above, is different from the proof in [8].

The argument above can also be extended to the corresponding Neumann boundary problem in two dimensions, whose viscosity solution theory was established in [41, 20]. Classical solutions of curvature flows with contact angle conditions were first considered by [23] and later by [1, 21, 42] etc. In Section 5, based on the game interpretation of Neumann boundary conditions in [19, 11], we use a similar method to prove the convexity preserving of superlevel sets of the viscosity solution under an extra assumption that the initial value u_0 is compatible with the boundary condition. We also give an example (Example 5.7), showing that the curvature flow may fail to preserve convexity if the compatibility condition is dropped.

The second part is on the concavity (convexity) preserving property of the viscosity solution u itself; see Theorem 1.2. We show, again by games, that $u(x,t)$ is concave in x in the support of $u_0 - a$ if u_0 is concave in the same set. The idea is to compare the usual game with a modified game. Roughly speaking, for any concave function $f \in C(\mathbb{R}^2)$ and $x, y \in \mathbb{R}^2$, we find

$$\frac{1}{2}\max_{|v|=1}\min_{b=\pm 1} f(x+\sqrt{2}\varepsilon bv) + \frac{1}{2}\max_{|v'|=1}\min_{b'=\pm 1} f(y+\sqrt{2}\varepsilon bv)$$
$$\leq \max_{|v|=|v'|=1}\min_{b,b'=\pm 1} f\left(\frac{x+y}{2} + \sqrt{2}\varepsilon\frac{bv+b'v'}{2}\right),$$

where the right hand side suggests that we consider a different game with averaged move of two choices for each player in every step. Denote by w^ε the value function of such a modified game.

By iterating the inequality with $f = u^\varepsilon(\cdot, j\varepsilon^2)$ for all $j = 1, 2, ...$, we get

$$u^\varepsilon(x,t) + u^\varepsilon(y,t) \leq 2w^\varepsilon\left(\frac{x+y}{2}, t\right)$$

for any $t \geq 0$ and $\varepsilon > 0$. However, on the other hand, it turns out that the upper relaxed limit $\overline{w}$ of w^ε, as $\varepsilon \to 0$, is still a subsolution of (MCF). This implies $\overline{w} \leq u$ by the usual comparison principle and concludes our game proof for convexity preserving.

We first found such a convexity argument for the solution of the parabolic normalized p-Laplace equation ($1 < p \leq \infty$), though it is not the main topic in this work; see more details in Section 2. The proof in this case is actually much simpler. By establishing a tug-of-war game similar to the one introduced in [36] (but in the whole space $\mathbb{R}^n$), one easily sees that the game value $u^\varepsilon(x,t)$ has the convexity in x for any $t > 0$ if it is convex initially. Its proof is only an application of the so-called marching argument proposed in [5], which is essentially an iteration of DPP; see also [35] and [33].

In contrast, our convexity proof for the equations of mean curvature type needs extra work. The main reason is that in the dynamic programming principles, the control set for each player is a solid ball in tug-of-war games for p-Laplacian ($p > 1$) while it appears only on a sphere in the game for mean curvature flow. The set convexity facilitates the proof of the convexity for the solution of p-Laplace equations ($p > 1$).

Finally, it is worthing mentioning that the discrete game interpretations of various elliptic and parabolic PDEs ([28, 38, 39, 29, 37, 36], etc) have recently attracted great attention. The game related methods are also used as a new tool in different contexts.

Armstrong and Smart [4] proved the uniqueness for infinity harmonic functions using a method related to the tug-of-war games in [38]. The fattening phenomenon for mean curvature flow is rigorously proved via games without using parabolic theory [32]. A recent work [34] provides a new proof of Harnack's inequality for p-Laplacian by stochastic games. All of these results largely simplify the original PDE proofs and indicate a strong potential of applicability of the game-theoretic approach.

This paper is organized in the following way. We first show the game-theoretic argument for convexity preserving in the case of tug-of-war games with and without noise in Section 2. In Section 3, we review the game setting for the mean curvature flow equation and provide a new modified game for comparison. We present game-theoretic proofs for convexity in Section 4. The convexity preserving for level sets is given in Section 4.1 and that for the solution itself is presented in Section 4.2. Section 5 is devoted to convexity preserving for motion by curvature in a planar domain with right contact angle.

2. Convexity Preserving for Normalized p-Laplacians

Our game-based arguments for convexity can be also applied to parabolic p-Laplacian equations ($2 \leq p \leq \infty$):

$$
\text{(PL)} \quad \begin{cases} u_t - \operatorname{tr}\left(\left(I + (p-2)\dfrac{\nabla u \otimes \nabla u}{|\nabla u|^2}\right)\nabla^2 u\right) = f(x) & \text{in } \mathbb{R}^n \times (0,\infty), \quad (2.1) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n, \quad (2.2) \end{cases}
$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a given continuous function.

It is expected, though not proved yet, that the equation is related to the DPP below:

$$
\begin{aligned} u^\varepsilon(x,t) = \frac{\alpha}{2} \sup_{v \in B_\varepsilon} u^\varepsilon(x+v, t-\varepsilon^2) + \frac{\alpha}{2} \inf_{v \in B_\varepsilon} u^\varepsilon(x+v, t-\varepsilon^2) \\ + \beta \fint_{v \in B_\varepsilon} u^\varepsilon(x+v, t-\varepsilon^2)\, dv + \varepsilon^2 f(x) \end{aligned} \tag{2.3}
$$

with

$$
u^\varepsilon(x,t) = u_0(x) \text{ for all } x \in \mathbb{R}^n \text{ and } t < \varepsilon^2, \tag{2.4}
$$

where $\fint_{y \in A} f(y)\, dy$ denotes the average value of $f \in C(\mathbb{R}^n)$ in $A \subset \mathbb{R}^n$. Here $\alpha \geq 0, \beta \geq 0$ satisfy $\alpha + \beta = 1$ and are determined by the choice of p. In fact, it is known from [36, 37] that

$$
\alpha = \frac{p-2}{p+n} \text{ and } \beta = \frac{2+n}{p+n}.
$$

One may show that u^ε converges, as $\varepsilon \to 0$, to the unique solution u of (PL) locally uniformly in $\mathbb{R}^n \times [0,\infty)$ by following [36].

We aim to prove

Theorem 2.1. *Assume that u_0 and f are both convex and Lipschitz continuous in $\mathbb{R}^n$. Suppose that the function u^ε satisfying (2.3)–(2.4) converges to the solution u of (PL). Then the solution $u(x,t)$ is convex in x for any $t \geq 0$; namely,*

$$
\frac{1}{2}u(x-h,t) + \frac{1}{2}u(x+h,t) \geq u(x,t)
$$

for any $x, h \in \mathbb{R}^n$ and $t \geq 0$.

Proof. To get the convexity preserving, it suffices to show that

$$\frac{1}{2}u^\varepsilon(x-h,t)+\frac{1}{2}u^\varepsilon(x+h,t)\geq u^\varepsilon(x,t).$$

Let us take a look at the first game step:

$$\begin{aligned} u^\varepsilon(x+h,\varepsilon^2) =&\frac{\alpha}{2}\sup_{v\in B_\varepsilon}u_0(x+h+v)+\frac{\alpha}{2}\inf_{v\in B_\varepsilon}u_0(x+h+v)\\ &+\beta \fint_{v\in B_\varepsilon}u_0(x+h+v)\,dv+\varepsilon^2 f(x+h),\\ u^\varepsilon(x-h,\varepsilon^2) =&\frac{\alpha}{2}\sup_{v\in B_\varepsilon}u_0(x-h+v)+\frac{\alpha}{2}\inf_{v\in B_\varepsilon}u_0(x-h+v)\\ &+\beta \fint_{v\in B_\varepsilon}u_0(x-h+v)\,dv+\varepsilon^2 f(x-h). \end{aligned} \tag{2.5}$$

We assume the infimum appearing in each equalities in (2.5) above is attained respectively at $v_1, v_2\in B_\varepsilon$ and also assume that $\sup_{v\in B_\varepsilon}u_0(x+v)=u_0(x+\hat{v})$ for some $\hat{v}\in B_\varepsilon$. (These can be rigorously justified by putting errors.) Hence, we have

$$\begin{aligned} u^\varepsilon(x+h,\varepsilon^2)\geq\ &\frac{\alpha}{2}u_0(x+h+\hat{v})+\frac{\alpha}{2}u_0(x+h+v_1)\\ &+\beta \fint_{v\in B_\varepsilon}u_0(x+h+v)\,dv+\varepsilon^2 f(x+h);\\ u^\varepsilon(x-h,\varepsilon^2)\geq\ &\frac{\alpha}{2}u_0(x-h+\hat{v})+\frac{\alpha}{2}u_0(x-h+v_2)\\ &+\beta \fint_{v\in B_\varepsilon}u_0(x-h+v)\,dv+\varepsilon^2 f(x-h). \end{aligned}$$

By convexity of u_0 and f, we have

$$\begin{aligned} &\frac{1}{2}u_0(x+h+\hat{v})+\frac{1}{2}u_0(x-h+\hat{v})\geq u_0(x+\hat{v});\\ &\frac{1}{2}u_0(x+h+v_1)+\frac{1}{2}u_0(x-h+v_2)\geq u_0(x+\frac{v_1+v_2}{2});\\ &\fint_{v\in B_\varepsilon}u_0(x+h+v)\,dv+\fint_{v\in B_\varepsilon}u_0(x-h+v)\,dv\geq 2\fint_{v\in B_\varepsilon}u_0(x+v)\,dv;\\ &f(x+h)+f(x-h)\geq 2f(x). \end{aligned}$$

Therefore, we sum up the inequalities and get

$$\begin{aligned} &u^\varepsilon(x+h,\varepsilon^2)+u^\varepsilon(x-h,\varepsilon^2)\\ &\geq \alpha u_0(x+\hat{v})+\alpha u_0(x+\frac{v_1+v_2}{2})+2\beta\fint_{v\in B_\varepsilon}u_0(x+v)\,dv+2\varepsilon^2 f(x)\\ &\geq \alpha\sup_{v\in B_\varepsilon}u_0(x+v)+\alpha\inf_{v,v'\in B_\varepsilon}u_0(x+\frac{v+v'}{2})+2\beta\fint_{v\in B_\varepsilon}u_0(x+v)\,dv+2\varepsilon^2 f(x)\\ &=\alpha\sup_{v\in B_\varepsilon}u_0(x+v)+\alpha\inf_{v\in B_\varepsilon}u_0(x+v)+2\beta\fint_{v\in B_\varepsilon}u_0(x+v)\,dv+2\varepsilon^2 f(x). \end{aligned} \tag{2.6}$$

Noting that by definition, the right hand side is just $2u^\varepsilon(x,\varepsilon^2)$, we have

$$u^\varepsilon(x+h,\varepsilon^2)+u^\varepsilon(x-h,\varepsilon^2)\geq 2u^\varepsilon(x,\varepsilon^2).$$

We repeat the argument up to $[\frac{t}{\varepsilon^2}]$ times and end up with

$$u^\varepsilon(x+h,t)+u^\varepsilon(x-h,t)\geq 2u^\varepsilon(x,t)$$

for any $t \geq 0$. Sending $\varepsilon \to 0$, we get the convexity desired. □

3. The Game Setting for Curvature Flow

Let us first review the game proposed in [28] for motion by curvature.

3.1. The game for mean curvature flow. A marker, representing the *game position* or *game state*, is initialized at $x \in \mathbb{R}^2$ from time 0. The maturity time given is denoted by t. Let the step size for space be $\varepsilon > 0$. Time ε^2 is consumed for each step. The total game steps N can be regarded as $[t/(\varepsilon^2)]$.

Two players, Player I and Player II participate the game: Player I intends to maximize at the final state an *objective function*, which in our case is u_0, while Player II is to minimize it. At each round,

(1) Player I chooses in $\mathbb{R}^2$ a unit vector v;
(2) Player II has the right to reverse the choice of Player I, which determines a sign $b = \pm 1$;
(3) The marker is moved from the present state x to $x + \sqrt{2}\varepsilon bv$.

To give a mathematical description, we denote

$$\mathbf{S}^1 = \{v \in \mathbb{R}^2 : |v| = 1\}.$$

Then the inductive *state equation* writes as

$$\begin{cases} z_{k+1} = z_k + \sqrt{2}\varepsilon b_k v_k, & k = 0, 1, \ldots, N-1; \\ z_0 = x, \end{cases}$$

where $v_k \in \mathbf{S}^1$ and $b_k = \pm 1$.

Hereafter, for any $x \in \mathbb{R}^n$ and $s \in [0, \infty)$, $z(s; x) = z_m$ stands for the game state at the step $m = [s/\varepsilon^2]$ starting from x under the competing strategies so that our games look like continuous ones.

Note that we here insist, for simplicity, considering the game in time period $[0, t]$ so that the associated equation is exactly (MC) instead of a backward-in-time one as in [28]. More precisely, the *value function* is defined as

$$u^\varepsilon(x, t) := \max_{v_1 \in \mathbf{S}^1} \min_{b_1 = \pm 1} \ldots \max_{v_N \in S^N} \min_{b_N = \pm 1} u_0(z(t; x)). \tag{3.1}$$

The *dynamic programming principle* follows easily:

$$u^\varepsilon(x, t) = \max_{v \in \mathbf{S}^1} \min_{b = \pm 1} u^\varepsilon(x + \sqrt{2}\varepsilon bv, t - \varepsilon^2), \tag{3.2}$$

which is employed to show the following theorem.

Theorem 3.1 (Theorem 1.2 in [28])**.** *Assume that $u_0 \in C(\mathbb{R}^2)$ satisfies (A). Let u^ε be the value function defined as in (3.1). Then u^ε converges, as $\varepsilon \to 0$, to the unique viscosity solution of (MCF) uniformly on compact subsets of $\mathbb{R}^2 \times (0, \infty)$.*

We recall the definition of viscosity solutions for (MCF) below.

Definition 3.2 ([16]). A locally bounded upper (resp., lower) semicontinuous function u is called a subsolution (resp., supersolution) of (1.1) if for any $(x_0, t_0) \in \mathbb{R}^2$ and $\phi \in C^2(\mathbb{R}^2 \times [0,\infty))$ such that $u - \phi$ attains a (strict) maximum (resp., minimum) at (x_0, t_0), we have

$$\phi_t - |\nabla\phi| \operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) \leq 0 \quad \text{at } (x_0, t_0),$$

$$\left(\text{resp.,} \quad \phi_t - |\nabla\phi| \operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) \geq 0 \quad \text{at } (x_0, t_0)\right)$$

when $\nabla\phi(x_0, t_0) \neq 0$ and

$$\phi_t(x_0, t_0) \leq 0, \quad (\text{resp.,} \quad \phi_t(x_0, t_0) \geq 0)$$

when $\nabla\phi(x_0, t_0) = 0$ and $\nabla^2\phi(x_0, t_0) = O$.

A locally bounded continuous function u is called a solution if it is both a subsolution and a supersolution.

A subsolution (resp., supersolution, solution) u of (1.1) is said to be a subsolution (resp., supersolution, solution) of (MCF) if it further satisfies $u(x, 0) \leq u_0(x)$ (resp., $u(x, 0) \geq u_0(x)$, $u(x, 0) = u_0(x)$) for all $x \in \mathbb{R}^2$.

3.2. **A modified game.** Let us slightly change the rules of the game described above: we keep the basic setting of the game and the objectives of both Players, but at each round, we now ask that

(1) Player I chooses $v, v' \in \mathbf{S}^1$;
(2) Player II determines $b = \pm 1$ and $b' = \pm 1$;
(3) The marker is moved from the present state x to $x + \frac{\sqrt{2}\varepsilon}{2}(bv + b'v')$.

Under the new rules above, we may define the value function

$$w^\varepsilon(x, t) := \max_{v_1, v_1' \in \mathbf{S}^1} \min_{b_1, b_1' = \pm 1} \ldots \max_{v_N, v_N' \in \mathbf{S}^1} \min_{b_N, b_N' = \pm 1} u_0(\tilde{z}(t; x)), \tag{3.3}$$

where $\tilde{z}(t; x) = \tilde{z}_N$ is the solution of the following state equation:

$$\begin{cases} \tilde{z}_{k+1} = \tilde{z}_k + \dfrac{\sqrt{2}\varepsilon}{2}(b_k v_k + b_k' v_k'), \quad k = 0, 1, \ldots, N-1; \\ z_0 = x, \end{cases}$$

It is clear that the new dynamic programming principle is

$$w^\varepsilon(x, t) = \max_{v, v' \in \mathbf{S}^1} \min_{b, b' = \pm 1} w^\varepsilon\left(x + \frac{\sqrt{2}\varepsilon}{2}(bv + b'v'), t - \varepsilon^2\right), \tag{3.4}$$

for any $t \geq \varepsilon^2$.

It is not difficult to see that w^ε is Lipschitz continuous in space at any time if u_0 is Lipschitz continuous, as is shown for u^ε in [28, Appendix B].

Proposition 3.3 (Lipschitz continuity preserving property). *Let w^ε be the value function associated to the modified game described above. If there exists $L > 0$ such that*

$$|u_0(x) - u_0(y)| \leq L|x - y| \text{ for any } x, y \in \mathbb{R}^2,$$

then $w^\varepsilon(x, t)$ satisfies

$$|w^\varepsilon(x, t) - w^\varepsilon(y, t)| \leq L|x - y| \text{ for any } x, y \in \mathbb{R}^2,\ t \geq 0 \text{ and } \varepsilon > 0. \tag{3.5}$$

Proof. We prove this result by induction. By (3.4), we have

$$w^\varepsilon(x,\varepsilon^2) = \max_{v,v'\in\mathbf{S}^1}\min_{b,b'=\pm 1} u_0\left(x+\frac{\sqrt{2}\varepsilon}{2}(bv+b'v')\right)$$
$$w^\varepsilon(y,\varepsilon^2) = \max_{v,v'\in\mathbf{S}^1}\min_{b,b'=\pm 1} u_0\left(y+\frac{\sqrt{2}\varepsilon}{2}(bv+b'v')\right).$$

Let v_0, v_0' be the maximizer in the first relation and b_0, b_0' be the minimizer (with respect to the choice of $v=v_0, v'=v_0'$) in the second. We have

$$\begin{aligned}&w^\varepsilon(x,\varepsilon^2)-w^\varepsilon(y,\varepsilon^2)\\ &\leq u_0\left(x+\frac{\sqrt{2}\varepsilon}{2}(b_0v_0+b_0'v_0')\right)-u_0\left(y+\frac{\sqrt{2}\varepsilon}{2}(b_0v_0+b_0'v_0')\right),\end{aligned}$$

which, in view of the Lipschitz continuity of u_0, implies that

$$w^\varepsilon(x,\varepsilon^2)-w^\varepsilon(y,\varepsilon^2)\leq L|x-y| \text{ for all } x,y\in\mathbb{R}^2 \text{ and } \varepsilon>0.$$

By inductively applying this argument and (3.4), we are led to

$$w^\varepsilon(x,t)-w^\varepsilon(y,t)\leq L|x-y| \text{ for all } x,y\in\mathbb{R}^2 \text{ and } \varepsilon>0.$$

Then (3.5) follows immediately by interchanging the roles of x and y. □

We next compare this modified game with the original game. In fact, we can show that the relaxed upper limit

$$\overline{w}(x,t)=\limsup_{\varepsilon\to 0}{}^{*}\, w^\varepsilon(x,t)$$

is a subsolution of (MCF).

Theorem 3.4 (Subsolution). *Assume that $u_0\in C(\mathbb{R}^2)$ satisfies (A). Let w^ε be the value functions defined as in (3.3). Then $\overline{w}$ as defined above is a subsolution of* (1.1) *with $\overline{w}(\cdot,0)\leq u_0$ in $\mathbb{R}^2$.*

We present a detailed proof of Theorem 3.4, since the game setting is quite different from the original games in [28]. The following elementary result, whose proof can be found in [18], is needed in our argument.

Lemma 3.5 (Lemma 4.1 in [18]). *Suppose p is a unit vector in $\mathbb{R}^2$ and X is a real symmetric 2×2 matrix. Then there exists a constant $M>0$ that depends only on the norm of X, such that for any unit vector $\xi\in\mathbb{R}^2$,*

$$|\langle Xp^\perp,p^\perp\rangle-\langle X\xi,\xi\rangle|\leq M|\langle\xi,p\rangle|, \tag{3.6}$$

where $p^\perp$ denotes a unit orthonormal vector of p.

Remark 3.6. By homogeneity, when p is not necessarily a unit vector, the relation (3.6) still holds provided $|\xi|=|p|$.

Proof of Theorem 3.4. 1. We first show that $\overline{w}$ is a subsolution of (1.1).

Let us assume that there exist $(x_0,t_0)\in Q=\mathbb{R}^2\times(0,\infty)$ and $\phi\in C^2(Q)$ such that

$$\overline{w}(x,t)-\phi(x,t)<\overline{w}(x_0,t_0)-\phi(x_0,t_0) \text{ for all } (x,t)\in Q.$$

Then by definition, there exists $(x_\varepsilon,t_\varepsilon)\in Q$ such that

$$(x_\varepsilon,t_\varepsilon)\to(x_0,t_0),\ w^\varepsilon(x_\varepsilon,t_\varepsilon)\to\overline{w}(x_0,t_0),$$

and

$$(w^\varepsilon - \phi)(x_\varepsilon, t_\varepsilon) \geq (w^\varepsilon - \phi)(x, t) - \varepsilon^3$$

for any (x, t) in a neighborhood of (x_0, t_0) with size independent of ε. By (3.4), we then have

$$\phi(x_\varepsilon, t_\varepsilon) \leq \max_{v, v' \in \mathbf{S}^1} \min_{b, b' = \pm 1} \phi\left(x_\varepsilon + \frac{\sqrt{2}\varepsilon}{2}(bv + b'v'), t_\varepsilon - \varepsilon^2\right) + \varepsilon^3.$$

Applying Taylor expansion, we are led to

$$\begin{aligned} \varepsilon^2 \phi_t \leq \max_{v, v' \in \mathbf{S}^1} \min_{b, b' = \pm 1} \Bigg\{ & \frac{\sqrt{2}\varepsilon}{2} \langle \nabla\phi, bv + b'v' \rangle \\ & + \frac{\varepsilon^2}{4} \langle \nabla^2\phi(bv + b'v'), (bv + b'v') \rangle \Bigg\} + o(\varepsilon^2) \text{ at } (x_\varepsilon, t_\varepsilon). \end{aligned} \tag{3.7}$$

We denote by I the right hand side evaluated at $(x_\varepsilon, t_\varepsilon)$. One may first compare $\langle \nabla\phi(x_\varepsilon, t_\varepsilon), v \rangle$ and $\langle \nabla\phi(x_\varepsilon, t_\varepsilon), v' \rangle$ to determine b or b'. For example, if

$$|\langle \nabla\phi(x_\varepsilon, t_\varepsilon), v \rangle| \leq |\langle \nabla\phi(x_\varepsilon, t_\varepsilon), v' \rangle|, \tag{3.8}$$

then we may choose b'_0 such that

$$-|\langle \nabla\phi(x_\varepsilon, t_\varepsilon), v' \rangle| = \langle \nabla\phi(x_\varepsilon, t_\varepsilon), b'_0 v' \rangle$$

and thus by (3.8)

$$\langle \nabla\phi, bv + b'_0 v' \rangle \leq 0 \text{ for either } b = \pm 1.$$

Therefore we have at $(x_\varepsilon, t_\varepsilon)$

$$\begin{aligned} & \min_{b, b' = \pm 1} \left\{ \frac{\sqrt{2}\varepsilon}{2} \langle \nabla\phi, bv + b'v' \rangle + \frac{\varepsilon^2}{4} \langle \nabla^2\phi(bv + b'v'), (bv + b'v') \rangle \right\} \\ & \leq \min_{b = \pm 1} \left\{ -\frac{\sqrt{2}\varepsilon}{2} |\langle \nabla\phi, bv + b'_0 v' \rangle| + \frac{\varepsilon^2}{4} \langle \nabla^2\phi(bv + b'_0 v'), (bv + b'_0 v') \rangle \right\}, \end{aligned}$$

and then

$$I \leq \max_{v, v'} \min_{b} \left\{ -\frac{\sqrt{2}\varepsilon}{2} |\langle \nabla\phi, bv + b'_0 v' \rangle| + \frac{\varepsilon^2}{4} \langle \nabla^2\phi(bv + b'_0 v'), (bv + b'_0 v') \rangle \right\} + o(\varepsilon^2).$$

Hereafter, we always assume (3.8) and keep the choice $b' = b'_0$.

Case A. Assume $\nabla\phi(x_0, t_0) \neq 0$, which in turn implies that $\nabla\phi(x_\varepsilon, t_\varepsilon) \neq 0$ for all $\varepsilon > 0$ small. We apply Lemma 3.5 and Remark 3.6 with

$$p = |bv + b'_0 v'| \frac{\nabla\phi(x_\varepsilon, t_\varepsilon)}{|\nabla\phi(x_\varepsilon, t_\varepsilon)|}, \quad \xi = bv + b'_0 v', \quad X = \nabla^2\phi(x_\varepsilon, t_\varepsilon),$$

and get

$$\begin{aligned} & \left| \langle \nabla^2\phi(bv + b'_0 v'), (bv + b'_0 v') \rangle - |bv + b'_0 v'|^2 \left\langle \nabla^2\phi \frac{\nabla^\perp\phi}{|\nabla\phi|}, \frac{\nabla^\perp\phi}{|\nabla\phi|} \right\rangle \right| \\ & \leq \frac{M|bv + b'_0 v'|}{|\nabla\phi|} |\langle \nabla\phi, bv + b'_0 v' \rangle| \end{aligned}$$

for any $v, v' \in \mathbf{S}^1$. Hence, we have

$$I \leq \varepsilon^2 \Delta_1^G \phi + \varepsilon\Phi \text{ at } (x_\varepsilon, t_\varepsilon) \tag{3.9}$$

where we denote

$$\Delta_1^G \phi = \left\langle \nabla^2\phi \frac{\nabla^\perp\phi}{|\nabla\phi|}, \frac{\nabla^\perp\phi}{|\nabla\phi|} \right\rangle = |\nabla\phi| \operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)$$

and

$$\Phi(x,t) = \max_{v,v'} \min_b \Big\{ -\frac{\sqrt{2}}{2}|\langle \nabla\phi(x,t), bv + b_0'v'\rangle| + \varepsilon\left(\frac{|bv+b_0'v'|^2}{4} - 1\right)\Delta_1^G\phi(x,t) \\ + \frac{\varepsilon M}{4|\nabla\phi(x,t)|}|bv+b_0'v'||\langle\nabla\phi(x,t), bv+b_0'v'\rangle|\Big\}. \tag{3.10}$$

Here we denote for any function $\phi(x,t) \in C^1(\mathbb{R}^2)$ class with $x = (x_1, x_2)$,

$$\nabla^\perp\phi = (-\partial\phi/\partial x_2, \partial\phi/\partial x_1).$$

We shall show that $\Phi(x_\varepsilon, t_\varepsilon) \leq 0$ for $\varepsilon > 0$ sufficiently small. Indeed, we discuss two different cases. If there exists a subsequence ε_k such that the maximum attains at v and v' (depending on ε_k) and as $k \to \infty$,

$$\text{either } |\langle\nabla\phi(x_{\varepsilon_k}, t_{\varepsilon_k}), v\rangle| \to c \text{ or } |\langle\nabla\phi(x_{\varepsilon_k}, t_{\varepsilon_k}), v'\rangle| \to c$$

for some $c > 0$, then it is clear that there exists $b = \pm 1$ also depending on ε_k satisfying

$$|\langle\nabla\phi(x_{\varepsilon_k}, t_{\varepsilon_k}), bv + b_0'v'\rangle| > c,$$

since

$$\max_b |\langle\nabla\phi(x,t), bv+b_0'v'\rangle| \geq \frac{1}{2}(|\langle\nabla\phi(x,t), v+b_0'v'\rangle| + |\langle\nabla\phi(x,t), -v+b_0'v'\rangle|) \\ = \max\{|\langle\nabla\phi(x,t), v\rangle|, |\langle\nabla\phi(x,t), b_0'v'\rangle|\}.$$

In view of (3.10), this yields immediately that

$$\limsup_{\varepsilon_k\to 0} \Phi(x_{\varepsilon_k}, t_{\varepsilon_k}) \leq -\frac{\sqrt{2}c}{2}.$$

The remaining case is that, for any $\varepsilon > 0$ small, the maximum of Φ is attained at v and v' such that

$$|\langle\nabla\phi(x_\varepsilon, t_\varepsilon), v\rangle| \to 0 \text{ and } |\langle\nabla\phi(x_\varepsilon, t_\varepsilon), v'\rangle| \to 0 \text{ as } \varepsilon \to 0.$$

We denote

$$\eta_\varepsilon = \nabla\phi(x_\varepsilon, t_\varepsilon)/|\nabla\phi(x_\varepsilon, t_\varepsilon)| \text{ and } \eta_\varepsilon^\perp = \nabla^\perp\phi(x_\varepsilon, t_\varepsilon)/|\nabla\phi(x_\varepsilon, t_\varepsilon)|.$$

Let us take a modulus of continuity ω such that

$$\max\{|\langle\eta_\varepsilon, v\rangle|, |\langle\eta_\varepsilon, v'\rangle|\} \leq \omega(\varepsilon).$$

Without loss of generality, we may assume

$$|b_0'v' - \eta_\varepsilon^\perp| \leq \omega(\varepsilon).$$

(Otherwise, we have

$$|b_0'v' + \eta_\varepsilon^\perp| \leq \omega(\varepsilon)$$

and apply the similar argument below.) Then we may pick b such that

$$|bv - \eta_\varepsilon^\perp| \leq \omega(\varepsilon).$$

It follows that under this choice,

$$\left|\frac{1}{4}|bv+b_0'v'|^2 - 1\right| = \frac{1}{4}|bv - b_0'v'|^2 \leq \frac{1}{2}(|bv-\eta_\varepsilon^\perp|^2 + |b_0'v' - \eta_\varepsilon^\perp|^2) \leq \omega^2(\varepsilon).$$

It follows from (3.10) that

$$\Phi(x_\varepsilon, t_\varepsilon) \leq C\varepsilon\omega^2(\varepsilon) + C\varepsilon\omega(\varepsilon)$$

for some $C > 0$ independent of ε.

By (3.9), we thus have

$$\varepsilon^2\phi_t \le \varepsilon^2\Delta_1^G\phi + o(\varepsilon^2) \quad \text{at } (x_\varepsilon, t_\varepsilon). \tag{3.11}$$

Dividing the inequality by ε^2 and sending $\varepsilon \to 0$, we obtain

$$\phi_t(x_0, t_0) \le \Delta_1^G\phi(x_0, t_0).$$

Case B. It remains to show the viscosity inequality

$$\phi_t(x_0, t_0) \le 0 \tag{3.12}$$

under the conditions that $\nabla\phi(x_0, t_0) = 0$ and $\nabla^2\phi(x_0, t_0) = O$. We need to discuss two sub-cases regarding the value $\nabla\phi(x_\varepsilon, t_\varepsilon)$.

Case B-1. Suppose that there exist a subsequence $\nabla\phi(x_{\varepsilon_k}, t_{\varepsilon_k}) \neq 0$ for k arbitrarily large. Then we may repeat the argument as in Case A and reach (3.11) again. Note that in the present case,

$$\nabla^2\phi(x_\varepsilon, t_\varepsilon) \to \nabla^2\phi(x_0, t_0) = O.$$

We therefore obtain (3.12) immediately.

Case B-2. If $\nabla\phi(x_\varepsilon, t_\varepsilon) = 0$ for all $\varepsilon > 0$, then it follows immediately from (3.7) that

$$\varepsilon^2\phi_t(x_\varepsilon, t_\varepsilon) \le C\varepsilon^2\|\nabla^2\phi(x_\varepsilon, t_\varepsilon)\| + o(\varepsilon^2),$$

for some $C > 0$ independent of ε, which yields (3.12).

2. We finally show that $\overline{w}(x, 0) \le u_0(x)$. To this end, we construct a game barrier at every fixed $x_0 \in \mathbb{R}^2$. It is easily seen that for any $\delta > 0$, there exits $K_\delta > 0$ such that

$$u_0(x) \le w_0(x) = u_0(x_0) + \delta + K_\delta|x - x_0|^2 \text{ for all } x \in \mathbb{R}^2.$$

We now play the games with objective function w_0 starting from any x in a neighborhood of x_0. In the first step, we have

$$\left|x - x_0 + \frac{\sqrt{2}\varepsilon}{2}(bv + b'v')\right|^2 = |x - x_0|^2 + \sqrt{2}\varepsilon\langle x - x_0, bv + b'v'\rangle + \frac{\varepsilon^2}{2}|bv + b'v'|^2.$$

Note that for any v, v', Player II can pick b' (or b) so that

$$\langle x - x_0, bv + b'v'\rangle \le 0$$

for any b (or any b'). This can be done by fixing b' such that $\langle b'v', x - x_0\rangle = -|\langle v', x - x_0\rangle|$ if $|\langle v', x - x_0\rangle| \ge |\langle v, x - x_0\rangle|$. Such a strategy yields

$$\left|x - x_0 + \frac{\sqrt{2}\varepsilon}{2}(bv + b'v')\right|^2 \le 2\varepsilon^2 + |x - x_0|^2$$

for any v, v'. Repeating this strategy of Player II regardless of choices by Player I, we get

$$w^\varepsilon(x, t) \le w_0(x) + 2K_\delta t.$$

This means $\overline{w}(x_0, 0) \le w_0(x_0) \le u_0(x_0) + \delta$. We conclude the proof by letting $\delta \to 0$.

□

The following result is an immediate consequence of Theorem 3.4.

Corollary 3.7 (Comparison theorem)**.** *Suppose that u is the viscosity solution of (MCF) and $\overline{w}$ is the upper relaxed limit of the game value w^ε associated to the modified game. Then $\overline{w} \le u$ in $\mathbb{R}^2 \times (0, \infty)$.*

The proof of this comparison theorem can be found in [10, 14].

4. Convexity Preserving Properties

4.1. **Convexity of level sets.** For future use, let D_c^0 and E_c^0 respectively denote the open and closed c-superlevel set of u_0 for any $c \geq a$; that is,

$$D_c^0 = \{x \in \mathbb{R}^n : u_0(x) > c\}; \quad E_c^0 = \{x \in \mathbb{R}^n : u_0(x) \geq c\}.$$

As a result, we have $D_c^0 \subset D_d^0$ and $E_c^0 \subset E_d^0$ for any $c \geq d$. Moreover, we have $D_c^0 = \bigcup_{d>c} E_d^0$.

The following lemma holds in any dimension.

Lemma 4.1 (Monotonicity). *Suppose that u_0 satisfies (A). Assume that the superlevel set E_c^0 for each $c \geq a$ is convex. Let u^ε be the value function associated to the game. Then*

$$u^\varepsilon(x,t) \leq u^\varepsilon(x,s) \text{ for all } x \in \mathbb{R}^n,\ t \geq s \geq 0 \text{ and } \varepsilon > 0. \tag{4.1}$$

In particular, the solution u is monotone in time, i.e., $u(x,t) \leq u(x,s)$ for all $x \in \mathbb{R}^n$, $t \geq s \geq 0$.

Proof. Let us fix $\varepsilon > 0$. We begin our proof by claiming that for every $x \in \mathbb{R}^n$ and any $v \in \mathbf{S}^1$, there exists $b = \pm 1$ such that

$$u_0\left(x + \sqrt{2}\varepsilon b v\right) \leq u_0(x). \tag{4.2}$$

Indeed, assume that for some $v \in \mathbf{S}^1$ and some $\lambda > u_0(x)$,

$$u_0\left(x - \sqrt{2}\varepsilon v\right),\ u_0\left(x + \sqrt{2}\varepsilon v\right) \geq \lambda.$$

By convexity of the superlevel set E_λ^0, $x = \frac{1}{2}(x - \sqrt{2}\varepsilon v) + \frac{1}{2}(x + \sqrt{2}\varepsilon v) \in E_\lambda^0$ which contradicts the assumption that $u_0(x) < \lambda$.

Now let us corporate the claim into our games. Notice that for any game position $z(s;x)$ from an arbitrary $x \in \mathbb{R}^2$ after time $s > 0$, depending on the choices $v_1, b_1, v_2, b_2, \ldots, v_N, b_N$ of both players ($N = [s/\varepsilon^2]$), Player II may use the strategy described above to ensure

$$\max_{v \in \mathbf{S}^1} \min_{b = \pm 1} u_0\left(z(s;x) + \sqrt{2}\varepsilon b v\right) \leq u_0(z(s;x)) \text{ for any } v \in \mathbf{S}^1.$$

By keeping applying this strategy from step N to $N' = [t/\varepsilon^2]$, we obtain

$$\max_{v_{N+1}} \min_{b_{N'+1}} \ldots \max_{v_{N'}} \min_{b'_N} u_0\left(z(s;x) + \sqrt{2}\varepsilon \sum_{i=N+1}^{N'} b_i v_i\right) \leq u_0(z(s;x)).$$

Taking the extrema on both sides of the inequality above over b_N, v_N, b_{N-1}, v_{N-1}, $\ldots$, b_1 and v_1 in order, we get (4.1) by definition. It follows immediately that $u(x,t) \leq u(x,s)$ for all $t \geq s$, by passing to the limit as $\varepsilon \to 0$ in (4.1) with application of Theorem 3.1.

□

Theorem 4.2 (Convexity of level sets). *Suppose that u_0 satisfies (A). Assume that each superlevel set E_c^0 ($c \geq a$) of u_0 is convex. Let u^ε be the value function associated to the game. Then every superlevel set of $u^\varepsilon(\cdot,t)$ is almost convex for any $t \geq 0$ and $\varepsilon > 0$ in* *the sense that there exists a modulus ω of continuity depending on u_0 such that for any $x, y \in \mathbb{R}^2$ and $t \geq 0$,*

$$u^\varepsilon\left(\frac{x+y}{2}, t\right) \geq c - \omega(\varepsilon) \tag{4.3}$$

provided that $u^\varepsilon(x,t) \geq c$ and $u^\varepsilon(y,t) \geq c$. In particular, all superlevel sets of the solution $u(\cdot, t)$ of (MCF) are convex for every $t \geq 0$.

Proof of Theorem 4.2. We assume $x \neq y$, since otherwise the statements are trivial. Since $u^\varepsilon(x,t) \geq c$ and $u^\varepsilon(y,t) \geq c$, we have $u^\varepsilon(x,s) \geq c$ and $u^\varepsilon(y,s) \geq c$ for $s \leq t$ as well in virtue of the monotonicity shown in Lemma 4.1. (In particular, $u_0(x) \geq c$ and $u_0(y) \geq c$.)

Then for any $s \leq t$, there must exist maximizing strategies $S^I_{x,s}$ and $S^I_{y,s}$ of Player I such that regardless of the choices of Player II, we have $u_0(z(s;x)) \geq c$ and $u_0(z(s;y)) \geq c$ if $S^I_{x,s}$ and $S^I_{y,s}$ are applied respectively in the games starting from x and y.

We next consider the game started from $(x+y)/2$. In this case, Player I has the following possible of move: he keeps choosing $v = (x-y)/|x-y|$ until the game position enters $B_{\sqrt{2}\varepsilon}(x)$ or $B_{\sqrt{2}\varepsilon}(y)$. Here $B_r(\xi)$ denotes the open ball centered at $\xi \in \mathbb{R}^n$ with radius r. Without loss of generality, suppose that Player II chooses to let $z(\tau; (x+y)/2) \in B_{\sqrt{2}\varepsilon}(x)$ after time $\tau (\leq t)$. Then Player I may use $S^I_{x,s}$ with $s = t - \tau$ to bring the game position to $\xi \in \mathbb{R}^n$, which depends on the response of Player II to $S^I_{x,s}$. However, the same strategy of Player II may send x to $z(s;x)$, which is in the $\sqrt{2}\varepsilon$-neighborhood of z. Since $u_0(z(s;x)) \geq c$, we get the following estimate:

$$u_0(\xi) \geq u_0(z(s;x)) - \omega_0(\sqrt{2}\varepsilon) \geq c - \omega_0(\sqrt{2}\varepsilon).$$

The remaining case is that Player II may choose to let the game position wander away from the neighborhoods of x and y. But in this case the final position η must still stay on the line segment between x and y and therefore

$$u_0(\eta) \geq c,$$

due to the facts that $u_0(x) \geq c$ and $u_0(y) \geq c$ and the assumption that superlevel sets of u_0 are convex.

Since each of game outcomes is just for one possible strategy of Player I and his optimal strategy should be even better, we end up with (4.3) with $\omega(x)$ set to be $\omega_0(\sqrt{2}x)$.

Now in order to prove the statement concerning u, we only need to take the limit using Theorem 3.1. More precisely, for any $\delta > 0$, there exists $\varepsilon > 0$ such that

$$u^\varepsilon(x,t) \geq c - \delta \text{ and } u^\varepsilon(y,t) \geq c - \delta.$$

Our argument above yields

$$u^\varepsilon\left(\frac{x+y}{2}, t\right) \geq c - \delta - \omega(\varepsilon).$$

Then letting $\varepsilon \to 0$ and $\delta \to 0$, we have

$$u\left(\frac{x+y}{2}, t\right) \geq c.$$

□

Remark 4.3. In the theorem above, we do not assume the concavity of u_0 itself but the convexity of its level sets. Our convexity result is therefore only for the convexity of level sets as well. Also, to study the convexity of level sets for a particular level, it is not restrictive to assume that all level sets of u_0 are convex curves. Note that changing the other level sets of u_0 will not affect the evolution of the particular level set in question [14, 10, 16].

Remark 4.4. An alternative method, without assuming the convexity of all level sets but strict convexity of the initial level set, is to investigate the convexity of the level sets of the solution to (SP) with Ω (strictly) convex. The stationary problem has a similar game-theoretic approximation [28] but the value function $u^\varepsilon(x)$ in this case is defined to be the optimized first exit time of the domain starting from $x \in \Omega$. It is not difficult to see that the argument in the proof of Theorem 4.2 applies exactly to this case as well. A new PDE proof for this problem is recently given in [8].

Our game-theoretic proof above for convexity of level sets seems to work only in two dimensions. We provide below a game interpretation for convexity preserving of the viscosity solution itself.

4.2. **Concavity preserving property of solutions.** We next show the convexity preserving property for the solution itself. Due to an opposite choice of orientation, we here discuss instead the equivalent concavity preserving property, which, under the assumption (A), means that the solution $u(x,t)$ of (MCF) is concave with respect to x in its superlevel set

$$D_a^t := \{x \in \mathbb{R}^n : u(x,t) > a\}$$

if u_0 is concave in K_a.

Theorem 4.5 (Concavity preserving of the solution)**.** *Suppose that $u_0 \in C(\mathbb{R}^2)$ satisfies (A). Assume that u_0 is concave in K_a. Let u^ε and w^ε be respectively the value functions of the original game and the modified game. Then $u^\varepsilon(x,t)$ satisfies*

$$u^\varepsilon(x+h,t) + u^\varepsilon(x-h,t) \leq 2w^\varepsilon(x,t), \tag{4.4}$$

for any $x, h \in \mathbb{R}^2$, $t > 0$ and $\varepsilon > 0$ if

$$u^\varepsilon(x+h,t) > a \text{ and } u^\varepsilon(x-h,t) > a.$$

Moreover, the solution $u(x,t)$ of (MCF) is concave with respect to x in D_a^t for any $t \geq 0$.

To show this result, we need the following elementary result related to our games.

Lemma 4.6. *Let Ω be a convex domain in $\mathbb{R}^2$. Suppose that $w, W \in C(\overline{\Omega})$ satisfy the following*

1. *$w, W > a$ in Ω for some $a \in \mathbb{R}$ and $w = W = a$ on $\partial\Omega$;*
2. *for all $x, h \in \mathbb{R}^n$ such that $x \pm h \in \Omega$,*

$$w(x+h) + w(x-h) \leq W(x). \tag{4.5}$$

Then for any constant $\lambda > 0$,

$$\begin{aligned} &\max_{v \in \mathbf{S}^1} \min_{b=\pm 1}\ w(x+h+\lambda bv) + \max_{v} \min_{b}\ w(x-h+\lambda bv) \\ &\quad \leq \max_{v^1, v^2 \in \mathbf{S}^1} \min_{b^1, b^2 = \pm 1} W\left(x + \frac{\lambda}{2}(b^1 v^1 + b^2 v^2)\right) \end{aligned} \tag{4.6}$$

provided that

$$\max_{v\in\mathbf{S}^1}\min_{b=\pm1}\ w(x+h+\lambda bv)>a \ \textit{and}\ \max_{v\in\mathbf{S}^1}\min_{b=\pm1}\ w(x-h+\lambda bv)>a. \tag{4.7}$$

Proof. We take v_+ and v_- such that

$$\max_v\min_b w(x\pm h+\lambda bv)=\min_b w(x\pm h+\lambda bv_\pm)(>a), \tag{4.8}$$

Let us next pick $b_\pm$ such that

$$W\left(x+\frac{\lambda}{2}(b_+v_++b_-v_-)\right)=\min_{b^1}\min_{b^2}W\left(x+\frac{\lambda}{2}(b^1v_++b^2v_+)\right),$$

which implies that

$$W\left(x+\frac{\lambda}{2}(v_+\cdot b_++v_-\cdot b_-)\right)\le\max_{v^1,v^2}\min_{b^1,b^2}W\left(x+\frac{\lambda}{2}(b_1v^1+b^2v^2)\right) \tag{4.9}$$

On the other hand, by (4.8), we have

$$\max_{v\in\mathbf{S}^1}\min_{b=\pm1}w(x\pm h+\lambda bv)\le w(x\pm h+\lambda b_\pm v_\pm). \tag{4.10}$$

Since by (4.10) and (4.7) we have

$$w(x\pm h+\lambda b_\pm v_\pm)>a,$$

we obtain (4.6) by combining (4.5), (4.9) and (4.10). □

We next apply this lemma to our games.

Proof of Theorem 4.4. Fix $\varepsilon>0$. We apply Lemma 4.6 with $\lambda=\sqrt{2}\varepsilon$, $\overline{\Omega}=K_a$ and

$$w(x)=u_0(x)\text{ and }W(x)=2u_0(x).$$

Since it is clear that (4.5) holds due to the concavity of u_0 in K_a, we get

$$\max_{v\in\mathbf{S}^1}\min_{b=\pm1}\ w(x+h+\sqrt{2}\varepsilon bv)+\max_{v\in\mathbf{S}^1}\min_{b=\pm1}\ w(x-h+\sqrt{2}\varepsilon bv)$$
$$\le\max_{v^1,v^2\in\mathbf{S}^1}\ \min_{b^1,b^2=\pm1}\ W\left(x+\frac{\sqrt{2}\varepsilon}{2}(b^1v^1+b^2v^2)\right)$$

if both terms on the left hand side are greater than a. This amounts to saying that

$$u^\varepsilon(x+h,\varepsilon^2)+u^\varepsilon(x-h,\varepsilon^2)\le2w^\varepsilon(x,\varepsilon^2) \tag{4.11}$$

for all x,h provided that $u^\varepsilon(x\pm h,\varepsilon^2)>a$. Noticing that $u^\varepsilon(x,\varepsilon^2)$ and $w^\varepsilon(x,\varepsilon^2)$ are uniformly continuous in x, due to Proposition 3.3 and [28, Appendix B], we can continue using Lemma 4.6 with $\lambda=\sqrt{2}\varepsilon$, $\Omega=D_a^{\varepsilon^2}$ and

$$w(x)=u^\varepsilon(x,\varepsilon^2)\text{ and }W(x)=2w^\varepsilon(x,\varepsilon^2).$$

Analogously, we obtain

$$u^\varepsilon(x+h,2\varepsilon^2)+u^\varepsilon(x-h,2\varepsilon^2)\le2w^\varepsilon(x,2\varepsilon^2)$$

if $u^\varepsilon(x\pm h,2\varepsilon^2)>a$, thanks to the dynamic programming principle:

$$u^\varepsilon(x+h,2\varepsilon^2)=\max_{v\in\mathbf{S}^1}\min_{b=\pm1}u^\varepsilon(x+h+\sqrt{2}\varepsilon bv,\varepsilon^2);$$
$$u^\varepsilon(x+h,2\varepsilon^2)=\max_{v\in\mathbf{S}^1}\min_{b=\pm1}u^\varepsilon(x+h+\sqrt{2}\varepsilon bv,\varepsilon^2);$$
$$w^\varepsilon(x,2\varepsilon^2)=\max_{v^1,v^2\in\mathbf{S}^1}\ \min_{b^1,b^2=\pm1}\ w^\varepsilon\left(x+\frac{\sqrt{2}\varepsilon}{2}(b^1v^1+b^2v^2),\varepsilon^2\right).$$

We keep iterating the arguments above and eventually get

$$u^\varepsilon(x+h, N\varepsilon^2) + u^\varepsilon(x-h, N\varepsilon^2) \leq 2w^\varepsilon(x, N\varepsilon^2), \tag{4.12}$$

for any $x, h \in \mathbb{R}^n$ as long as $u^\varepsilon(x+h, N\varepsilon^2) > a$ and $u^\varepsilon(x-h, N\varepsilon^2) > a$. This is exactly the desired inequality (4.4) when $N = [t/\varepsilon^2]$.

Since $\varepsilon > 0$ is arbitrary in (4.12), by passing to the limits as $\varepsilon \to 0$ and applying Theorem 3.1, we get

$$u(x+h, t) + u(x-h, t) \leq 2\overline{w}(x, t)$$

provided that (4.7) holds.

The concavity preserving property for the solution u follows immediately from the comparison that $\overline{w} \leq u$ in Corollary 3.7. □

The concavity or convexity preserving property does not precisely hold on the discrete level in general. In other words, one cannot expect in general that

$$u^\varepsilon(x-h, t) + u^\varepsilon(x+h, t) \leq 2u^\varepsilon(x, t)$$

for all $x, h \in \mathbb{R}^2$, $t > 0$ and $\varepsilon > 0$. We give an example to show this.

Example 4.7. Suppose that $u_0 \in C(\mathbb{R}^2)$ is concave and the level set $\{x \in \mathbb{R}^2 : u_0(x) = 0\}$ contains the positive axes close to the origin, say $\{0\} \times [0,1] \cup [0,1] \times \{0\}$. We assume that $u_0 > 0$ in $[0,1] \times [0,1]$. Let $x_0 = (\frac{\sqrt{2}}{2}\varepsilon, \frac{\sqrt{2}}{2}\varepsilon)$ and $h = (\frac{\sqrt{2}}{2}\varepsilon, -\frac{\sqrt{2}}{2}\varepsilon)$.

When $\varepsilon > 0$ is taken small, it is easy to see that $u^\varepsilon(x_0+h, \varepsilon^2) = u^\varepsilon(x_0-h, \varepsilon^2) = 0$, since the maximizing choices of v for Player I to start the game at x_0+h or x_0-h are the ones along the axes. On the other hand, at the point x_0, the outcome after one step is always negative no matter what choice Player I makes, i.e., $u^\varepsilon(x_0, \varepsilon^2) < 0$. We therefore have

$$u^\varepsilon(x_0+h, \varepsilon^2) + u^\varepsilon(x_0-h, \varepsilon^2) \geq 2u^\varepsilon(x_0, \varepsilon^2),$$

although u_0 itself is concave.

One may also easily see that, when $\varepsilon > 0$ is sufficiently small, the zero level set $\{x \in \mathbb{R}^2 : u^\varepsilon(x, \varepsilon^2) = 0\}$ contains the piece

$$\{0\} \times \left[\sqrt{2}\varepsilon, \frac{1}{2}\right] \cup \left[\sqrt{2}\varepsilon, \frac{1}{2}\right] \times \{0\} \cup \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 2\varepsilon^2, x_1, x_2 > 0\},$$

which is clearly not convex.

5. Convexity Preserving with Neumann boundary condition

Our game-theoretic approach to convexity can be extended to the Neumann problems as well. In this section, we assume that Ω is a smooth bounded convex domain in $\mathbb{R}^2$ and $\nu(x)$ denote the unit outward normal to $\partial\Omega$. We consider the Neumann boundary problem for the level set curvature flow equation in Ω:

$$\text{(NP)} \quad \begin{cases} u_t - |\nabla u| \operatorname{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) = 0 & \text{in } \Omega \times (0, \infty), \quad (5.1) \\ \nabla u(x,t) \cdot \nu(x) = 0 & \text{for } x \in \partial\Omega \text{ and } t > 0, \quad (5.2) \\ u(x, 0) = u_0(x) & \text{for all } x \in \overline{\Omega}. \quad (5.3) \end{cases}$$

Hereafter we assume an analogue of (A):

(A1) u_0 is a bounded and uniformly continuous function in $\overline{\Omega}$ satisfying that $u_0 \geq a$ and $u_0 - a$ has compact support $K_a \subset \overline{\Omega}$ for some constant $a \in \mathbb{R}$.

We refer [20, 41] for existence and uniqueness for viscosity solutions of this problem.

5.1. **Billiard games for the Neumann condition.** We first recall a billiard game interpretation for this problem in [19]. For any $t \geq 0$, $x \in \overline{\Omega}$ and $v \in \mathbf{S}^1$, let $S^t(x, v) \in \overline{\Omega}$ denote the position of billiard motion starting from x with initial direction v and with distance of travel equal to t; see [19, Definition 2.1].

In view of [19, Lemma 2.3], we have

$$S^t(x, v) = x + tv - \alpha^t(x, v), \tag{5.4}$$

where $\alpha^t(x, v) \in \mathbb{R}^2$ is called a boundary adjustor, satisfying $|\alpha^t(x, v)| \leq 2t$. More precisely, we may write

$$\alpha^t(x, v) = \sum_{k=0}^{\infty} c_k \nu(y_k), \tag{5.5}$$

with $c_k \in \mathbb{R}$, $y_k \in \partial\Omega \cap B_t(x)$.

We apply this billiard dynamics to our game setting by slightly changing the game rules introduced in Section 3.1. We restrict the starting point x to be in $\overline{\Omega}$ and substitute the rule (3) with the following:

(3)' The marker is moved from the present state x to $S^{\sqrt{2}\varepsilon}(x, bv)$.

We may define the value function u^ε to be in the same form of (3.1) by establishing a new state equation:

$$\begin{cases} z_{k+1} = S^{\sqrt{2}\varepsilon}(z_k, b_k v_k), \quad k = 0, 1, \ldots, N-1; \\ z_0 = x, \end{cases} \tag{5.6}$$

where $v_k \in \mathbf{S}^1$ and $b_k = \pm 1$.

Then u^ε converges to the unique viscosity solution of (NP).

Theorem 5.1. [19, Theorem 1.1] *Assume that Ω is a smooth bounded convex domain in $\mathbb{R}^2$. Let u^ε be the value function associated to the game above. Then u^ε converges, as $\varepsilon \to 0$, to the unique viscosity solution of (NP) uniformly on compact subsets of $\overline{\Omega} \times [0, \infty)$.*

Remark 5.2. There is more than one way to generate the boundary condition of Neumann type in games. One may simply adopt direct constraints of game trajectories in the domain to establish tug-of-war game interpretations of Neumann boundary problem for infinity Laplacian in [9, 3]. Their method relies on the special structure of the infinity Laplacian and is not applicable in our present case. A different but more relevant approach is presented in [11] for the Neumann problems for a general class of elliptic and parabolic equations, where the author uses boundary projection instead of our billiard motion. The game convergence result in [11] does not include (NP), but it is possible to adapt the argument to our case.

More precisely, when $\Omega \subset \mathbb{R}^2$ is sufficiently smooth, there exists $\varepsilon_0 > 0$ such that the projection $\mathrm{proj}_{\overline{\Omega}}(x)$ of any x in the ε_0-neighborhood of Ω onto $\overline{\Omega}$ is uniquely defined. We set for any $x \in \overline{\Omega}$, $v \in \mathbf{S}^1$ and $t \in [0, \varepsilon_0]$,

$$S_p^t(x, v) = \mathrm{proj}_{\overline{\Omega}}(x + tv). \tag{5.7}$$

Replacing $S^{\sqrt{2}\varepsilon}(z_k, b_k v_k)$ with $S_p^{\sqrt{2}\varepsilon}(z_k, b_k v_k)$, we may define a new value function u_p^ε. One can show u_p^ε also locally uniformly converges to the solution u of (NP) as $\varepsilon \to 0$ without assuming the boundedness and convexity of Ω but instead uniform interior and exterior conditions [11]. The proof is almost the same as that of Theorem 5.1, since an analogue of (5.4) and (5.5) holds: for any $t \in [0, \varepsilon_0]$

$$S_p^t(x, v) = \mathrm{proj}_{\overline{\Omega}}(x + tv) = x + tv - \tau\nu(S_p^t(x, v)),$$

for some $\tau \in [0, t]$.

5.2. **Convexity of level sets.** As in the preceding sections, we take for any $t \geq 0$

$$D_c^t = \{x \in \overline{\Omega} : u(x, t) > c\}, E_c^0 = \{x \in \overline{\Omega} : u(x, t) \geq c\} \text{ and } \Gamma_c^t = E_c^t \setminus D_c^t.$$

We aim to show that for any fixed $c \geq a$, E_c^0 being convex implies that E_c^t is convex for any $t \geq 0$. We however assume

(A2) E_c^0 is convex for any $c \geq a$.

It is clear that under the assumption (A2), for any $x \in \overline{\Omega}$ with $u_0(x) = c \geq a$, there exists a supporting line L_x of E_c^0 passing through x; that is, L_x divides $\mathbb{R}^2$ into two half planes, only one of which has nonempty intersection with D_c^0. We denote by $\xi(x)$ the outward unit normal to the half plane containing D_c^0.

We will also use a compatibility condition as given below:

Definition 5.3 (Compatibility condition)**.** For any u_0 satisfying (A2), we say u_0 is (weakly) compatible with the Neumann boundary condition (5.2) if for any $\tau > 0$ small and $x_0 \in \partial\Omega$ there exists a supporting line L_{x_0} with normal $\xi(x_0)$ satisfying

$$u_0(S^\tau(x_0, v)) \leq u_0(x_0)$$

for all $v \in \mathbf{S}^1$ with $\langle v, \xi(x_0)\rangle \geq 0$.

Theorem 5.4 (Convexity preserving for the Neumann problem)**.** *Assume that Ω is a smooth bounded convex domain in $\mathbb{R}^2$. Let u be the unique viscosity solution of (NP) with u_0 satisfying (A1) and (A2). Assume that u_0 also satisfies the compatibility condition as in Definition 5.3. Then the superlevel sets E_c^t of $u(\cdot, t)$ for any $t \geq 0$ are convex for any $t \geq 0$.*

In order to prove this theorem, we again need a monotonicity result similar to Lemma 4.1.

Lemma 5.5 (Monotonicity for the Neumann problem)**.** *Suppose that u_0 satisfies (A1) and (A2). Assume that u_0 also satisfies the compatibility condition. Let u^ε be the value function associated to the billiard game described above. Then*

$$u^\varepsilon(x, t) \leq u^\varepsilon(x, s) \text{ for all } x \in \mathbb{R}^n,\ t \geq s \geq 0 \text{ and } \varepsilon > 0. \tag{5.8}$$

In particular, the solution u is monotone in time, i.e., $u(x, t) \leq u(x, s)$ for all $x \in \mathbb{R}^n$, $t \geq s \geq 0$.

Proof. Let us fix $\varepsilon > 0$. We claim this time that for every $x \in \mathbb{R}^n$,

$$\sup_{v \in \mathbf{S}^1} \inf_{b = \pm 1} u_0\left(S^{\sqrt{2}\varepsilon}(x, bv)\right) \leq u_0(x). \tag{5.9}$$

Indeed, for any $v \in \mathbf{S}^1$, one may pick $\hat{b} = \pm 1$ such that $\langle \xi(x), \hat{b}v \rangle \geq 0$. If the billiard motion does not touch $\partial\Omega$ in the initial direction $\hat{b}v$, then by convexity of the superlevel set with level $u_0(x)$, we get

$$u_0\left(S^{\sqrt{2}\varepsilon}(x, \hat{b}v)\right) \leq u_0(x).$$

In the case that the billiard trajectory hits $\partial\Omega$, we have

$$S^{\sqrt{2}\varepsilon}(x, \hat{b}v) = S^{\sqrt{2}\varepsilon-\tau}(y, \hat{b}v), \tag{5.10}$$

where $y = x + \tau\hat{b}v \in \partial\Omega$ denotes the first hitting position with $\tau \geq 0$. For the same reason as above, y must belong to the lower level of u_0, i.e., $u_0(y) \leq u_0(x)$. We next apply the compatibility condition to get

$$u_0(S^{\sqrt{2}\varepsilon-\tau}(y, \hat{b}v)) \leq u_0(y),$$

which by (5.10) yields

$$u_0(S^{\sqrt{2}\varepsilon}(x, \hat{b}v)) \leq u_0(x).$$

The rest of the proof, similar to that in the proof of Lemma 4.1 consists in the iteration of (5.9) and passing to the limit as $\varepsilon \to 0$ with application of Theorem 5.1. We omit it and refer the reader to the proof of Lemma 4.1.

□

The proof of Theorem 5.4 is essentially the same as that of Theorem 4.2. However, in the proof of Theorem 4.2 we used the Lipschitz continuous dependence of game strategies on the initial positions and directions, which is not clear in the current case for the billiard motion. We thus provide a slightly different proof by choosing a subsequence of game values. This proof also works for Theorem 4.2.

Proof of Theorem 5.4. We take $x, y \in \overline{\Omega}$ with $x \neq y$ and $u(x,t), u(y,t) \geq c$ for some $t \geq 0$ and $c \geq a$. We aim to show that $u(\frac{x+y}{2}, t) \geq c$.

We set $\varepsilon_m = \frac{|x-y|}{2\sqrt{2}m}$ for any positive integer m. In view of Theorem 5.1, for any $\delta > 0$ small, there exists m sufficiently large, such that

$$u^{\varepsilon_m}(x,t), u^{\varepsilon_m}(y,t) \geq c - \delta.$$

We then have $u^\varepsilon(x,s) \geq c - \delta$ and $u^\varepsilon(y,s) \geq c - \delta$ for $s \leq t$ as well due to Lemma 5.5.

Then for any $s \leq t$, there must exist maximizing strategies $S^I_{x,s}$ and $S^I_{y,s}$ of Player I such that regardless of the choices of Player II, we have $u_0(z(s;x)) \geq c - 2\delta$ and $u_0(z(s;y)) \geq c - 2\delta$ if $S^I_{x,s}$ and $S^I_{y,s}$ are applied respectively in the games starting from x and y.

We next consider the game started from $(x+y)/2$. If Player I keeps choosing $v = (x-y)/|x-y|$ until the game position reaches x or y. Without loss of generality, suppose that Player II chooses to let $z(\tau; (x+y)/2) = x$ after time $\tau (\leq t)$. Then Player I may use $S^I_{x,s}$ with $s = t - \tau$ to bring the game position to $\xi \in \mathbb{R}^n$, which depends on the response of Player II to $S^I_{x,s}$. This yields

$$u_0(z(t; (x+y)/2)) = u_0(z(s;x)) \geq c - 2\delta.$$

Player II may choose to let the game position wander away from the neighborhoods of x and y. In this case the final position η must still stay on the line segment between x and y and therefore

$$u_0(\eta) \geq c - 2\delta,$$

due to the assumption that superlevel sets of u_0 are convex.

Since the above game estimate is for a fixed strategy of Player I, we get

$$u^{\varepsilon_m}\left(\frac{x+y}{2}, t\right) \geq c - 2\delta.$$

Thanks to Theorem 5.1, we conclude the proof by passing to the limit as $m \to \infty$ and then $\delta \to 0$. □

Remark 5.6. As mentioned in Remark 5.2, it is possible to drop the boundedness and convexity assumptions on Ω but keep the same game approximation result (Theorem 5.1) if S^t is replaced by S^t_p in the game. Therefore one may also expect that the convexity preserving property still holds without assuming the boundedness and convexity of Ω. In fact, without convexity of Ω, we can use the same argument to prove that E^t_c preserves convexity relative to Ω under the assumption (A1), (A2) and the compatibility condition. To be more precise, we have

$$u\left(\frac{x+y}{2}, t\right) \geq c$$

whenever $x, y \in \overline{\Omega}$ and $t \geq 0$ satisfy $u(x,t), u(y,t) \geq c$ and $kx + (1-k)y \in \overline{\Omega}$ for all $k \in [0,1]$.

We finally make some remarks on the compatibility condition in Definition 5.3.

Let us discuss a smooth special case. Suppose u_0 is of class C^2 and concave in $\overline{\Omega}$. Then u_0 is compatible with the Neumann boundary condition if there is $\sigma > 0$ such that

$$\langle \nabla u_0(x_0), \nu(y)\rangle \geq 0 \text{ and } \nabla^2 u_0(x_0) \leq -\sigma I. \tag{5.11}$$

for any $x_0 \in \partial\Omega$ and any $y \in B_\sigma(x_0) \cap \partial\Omega$.

Indeed, in this case one may choose $\xi(x_0) = -\nabla u_0(x_0)/|\nabla u_0(x_0)|$. To simplify notation, we write ξ_0 instead of $\xi(x_0)$. For any $v \in \mathbf{S}^1$ with $\langle v, \xi_0\rangle \geq 0$, we write

$$v = \tau_1 \xi_0 + \tau_2 \xi_0^\perp,$$

where $\xi_0^\perp$ is the unit vector orthogonal to ξ_0 and $\tau_1 \geq 0, \tau_2 \in \mathbb{R}$ satisfy $\tau_1^2 + \tau_2^2 = \tau^2$.

We apply Taylor expansion to get

$$\begin{aligned} u_0(S^\tau(x_0, v)) &= u_0(x_0) + \tau_1\langle \nabla u_0(x_0), \xi_0\rangle - \langle \nabla u_0(x_0), \alpha^\tau(x_0, v)\rangle \\ &\quad + \frac{1}{2}\langle \nabla^2 u_0(x_0)(\tau v - \alpha^\tau(x_0,v)), (\tau v - \alpha^\tau(x_0, v))\rangle + o(|\tau v - \alpha^\tau(x_0,v)|^2), \end{aligned}$$

which, due to (5.11) and (5.5), implies

$$u_0(S^\tau(x_0, v)) \leq u_0(x_0)$$

for τ sufficiently small.

It is worth pointing out that the curvature flow Γ_t fails to preserve convexity in general if the compatibility condition as in Definition 5.3 is not satisfied.

Example 5.7. Consider the special case when $\Omega = (-1, 1) \times \mathbb{R}$ and Γ_t can be expressed as the graph of a function $y = v(x, t)$ for $(x, t) \in [-1, 1] \times [0, \infty)$, we deduce

$$\begin{cases} v_t - \dfrac{v_{xx}}{1 + v_x^2} = 0 & \text{in } (-1,1) \times (0, \infty), & (5.12) \\ v_x(-1, t) = v_x(1, t) = 0 & \text{for all } t > 0, & (5.13) \\ v(x, 0) = v_0(x) & \text{for all } x \in [-1, 1]. & (5.14) \end{cases}$$

It is known [31, 6, 7] that there still exists a unique viscosity solution of this problem for any Lipschitz continuous function even when v_0 does not fulfill the compatibility condition. Suppose that v_0 is an even Lipschitz function on $[-1, 1]$ with

$$(v_0)_x(1) = -(v_0)_x(-1) > 0.$$

Then for any $t \geq 0$, the unique solution $v(x, t)$ must also be even in x.

On the other hand, one may extend v_0 to a Lipschitz function on $\mathbb{R}$ in a periodic manner and then solve the corresponding Cauchy problem in $\mathbb{R}\times[0, \infty)$. It is clear that the solution is still space periodic for any time. Moreover, it is shown by Ecker and Huisken [13] that the solution $\tilde{v}(x, t)$ is smooth for any $t > 0$, which implies that $\tilde{v}_x(\pm 1, t) = 0$ and $\tilde{v}$ cannot be convex around $x = \pm 1$. This means that the restriction of $\tilde{v}$ on $[-1, 1] \times [0, \infty)$ is the unique solution of (5.12)–(5.14). The failure of being convex near $x = \pm 1$ remains with such a restriction.

For the same reason as explained above, one cannot in general expect the viscosity solution of (NP) itself to be convexity preserving without assuming a more restrictive compatibility condition.

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