

ε -regularity for systems involving non-local, antisymmetric operators

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We prove an epsilon-regularity theorem for critical and super-critical systems with a non-local antisymmetric operator on the right-hand side.

These systems contain as special cases, Euler-Lagrange equations of conformally invariant variational functionals as Rivière treated them, and also Euler-Lagrange equations of fractional harmonic maps introduced by Da Lio-Rivière.

In particular, the arguments presented here give new and uniform proofs of the regularity results by Rivière, Rivière-Struwe, Da-Lio-Rivière, and also the integrability results by Sharp-Topping and Sharp, not discriminating between the classical local, and the non-local situations.

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1. Introduction

In recent years there has been quite some research on the effect of antisymmetric potentials in the regularity theory of critical and super-critical elliptic partial differential equations. This was initiated by Rivière who in his

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celebrated [Riv07] proved that solutions $u \in W^{1,2}(D, \mathbb{R}^N)$ to the equation

$$\Delta u = \Omega \cdot \nabla u \quad \text{in } D \subset \mathbb{R}^2, \quad (1.1)$$

which is a contracted notation of

$$\Delta u^i = \sum_{k=1}^N \Omega_{ik} \cdot \nabla u^k \quad 1 \leq i \leq N, \text{ in } D \subset \mathbb{R}^2,$$

are Hölder continuous, under the condition that $\Omega_{ij} \in L^2(D, \mathbb{R}^2)$ and the at first sight seemingly non-descript condition

$$\Omega_{ik} = -\Omega_{ki}, \quad 1 \leq i, k \leq N. \quad (1.2)$$

As Rivière showed, (1.1) with (1.2) is essentially the general form of Euler-Lagrange equations of conformally invariant variational functionals which allow the characterization of Grüter [Grü84], take for example a manifold $\mathcal{N} \subset \mathbb{R}^N$ and the Dirichlet energy

$$\int_{\mathbb{R}^2} |\nabla u|^2, \quad u : D \subset \mathbb{R}^2 \rightarrow \mathcal{N} \subset \mathbb{R}^N.$$

We refer the interested reader to the introduction of [Riv07] for more details. In [RS08] this was generalized to an epsilon-regularity theorem for $D \subset \mathbb{R}^m$, $m \geq 3$.

If the antisymmetry-condition (1.2) is violated, solutions to (1.1) might exhibit singularities such as Frehse's [Fre73] counter-example $\log \log \frac{1}{|x|}$. In fact, the antisymmetry is shown to be closely related to the appearance of Hardy spaces, and also to Hélein's [Hél91] moving frame technique, cf. [Sch10a].

Motivated by this, Da Lio and Rivière [DLR11a] (for $m = 1$) showed that this regularizing effect of antisymmetry exists and appears also in the setting of $m/2$ -harmonic maps, critical points of the energy

$$\int_{\mathbb{R}^m} \left| |\nabla|^{\frac{m}{2}} u \right|^2, \quad u : \mathbb{R}^m \rightarrow \mathcal{N} \subset \mathbb{R}^N.$$

which satisfy (roughly) an Euler-Lagrange equation of the form

$$\Delta^{\frac{m}{2}} u^i = \sum_{k=1}^N \Omega_{ik} |\nabla|^{\frac{m}{2}} u^k \quad 1 \leq i \leq N, \text{ in } D \subset \mathbb{R}^m. \quad (1.3)$$

Here, $\Omega_{ij} \in L^2(\mathbb{R}^m)$ satisfies again (1.2), and $|\nabla|^\alpha = (-\Delta)^{\frac{\alpha}{2}}$ is the elliptic differential operator of differential order α with the symbol $|\xi|^\alpha$, for the precise definition we refer to Section A.

As well in the classical situation [Riv07], as also in the case of fractional harmonic maps, the argument relies on transforming the equation with an orthogonal matrix P (in a similar way as Hélein's moving frame technique, cf. [Sch10a]). That is, one computes the respective equation $P\nabla u$ instead of ∇u , or $P\Delta^{\frac{m}{2}} u$ instead of $\Delta^{\frac{m}{2}} u$ and obtains a transformed Ω_P , which for the right choice of P exhibits better properties than the original Ω : In the classical case, $\text{div}(\Omega_P) = 0$, while in the fractional case, $\Omega_P \in L^{2,1}$ (where $L^{2,1} \subsetneq L^2$ is the Lorentz space dual to the weak L^2 , denoted by $L^{2,\infty}$). Note that while a condition like $\text{div}(f) = 0$ is destroyed under a distortion like $\tilde{f} := fg$, even for $g \in L^\infty$, the condition $f \in L^{2,1}$ is also valid for $\tilde{f} = fg$, if $g \in L^\infty$.

Thus, the techniques developed in the fractional setting [DLR11a, DLR11b, Sch11, DL10, Sch12], seem somewhat more dynamic and stable under certain distortions. For example, in [DLS12a, DLS12b] Da Lio and the author were able to extend some of the results to the degenerate situation of the energy

$$\int_{\mathbb{R}^m} \left\| |\nabla|^\alpha u \right\|^{\frac{m}{\alpha}}, \quad u : \mathbb{R}^m \rightarrow \mathcal{N} \subset \mathbb{R}^N,$$

the Euler-Lagrange equation of which have the form

$$|\nabla|^\alpha \left(\left\| |\nabla|^\alpha u \right\|^{\frac{m}{\alpha}-2} |\nabla|^\alpha u \right) = \left\| |\nabla|^\alpha u \right\|^{\frac{m}{\alpha}-2} \sum_{k=1}^N \Omega_{ik} |\nabla|^\alpha u^k \quad 1 \leq i \leq N, \text{ in } D \subset \mathbb{R}^m.$$

The aim of the present work is to shed more light on the connection between the two systems (1.3) and (1.1) in the critical and supercritical case, and we are going to extend the techniques developed in [DLR11a, DLR11b, Sch11, Sch12] to give a uniform argument for ε -regularity for quite general systems which in particular include as special

cases both (1.3) and (1.1).

Setting $w := (-\Delta)^{\frac{1}{2}}u \equiv |\nabla|^1 u \in L^2(\mathbb{R}^m)$, (1.1) reads as

$$\Delta^{\frac{1}{2}}w^i = \sum_{\gamma=1}^2 \sum_{k=1}^N \Omega_{ik}^\gamma \mathcal{R}_\gamma[w^k], \quad (1.4)$$

where $\mathcal{R}_\gamma \equiv \partial_\gamma \Delta^{-\frac{1}{2}}$ denotes the Riesz transform. Thus, (1.1) is of the form (1.3), but Ω is not a pointwise matrix anymore, but a non-local, linear operator mapping $L^2(\mathbb{R}^m)$ into $L^1(\mathbb{R}^m)$. This was our main motivation, to study the regularity, and, in the super-critical regime, ε -regularity of solutions $w \in L^2(\mathbb{R}^m)$ of

$$\int w_i |\nabla|^\mu \varphi = - \int \Omega_{ik}[w_k] \varphi \quad \text{for all } \varphi \in C_0^\infty(D), \quad (1.5)$$

where Ω_{ik} is a linear mapping which maps $L^2(\mathbb{R}^m)$ into $L^1(\mathbb{R}^m)$. We will restrict ourselves to Ω of the form

$$\Omega_{ij}[] = \sum_{l=0}^m A_{ij}^l \mathcal{R}_l[], \quad (1.6)$$

where $A_{ij}^l = -A_{ji}^l \in L^2(\mathbb{R}^m)$, $i, j \in 1, \dots, m$, $\mathcal{R}_l[]$ is the l -th Riesz transform for $l = 1, \dots, m$ and $\mathcal{R}_0[]$ is the identity on \mathbb{R}^m . The arguments presented here hold also for more general potentials $\Omega : L^2 \rightarrow L^1$, under suitable conditions on quasi-locality and its commutators. But as (1.6) contains already the most interesting examples (see below), we shall restrict our attention to this setting for the sake of overview.

Our main result is then the following ε -regularity:

Theorem 1.1. *Let $\mu \leq \min\{1, \frac{m}{2}\}$ or $\mu = \frac{m}{2}$. Let $D \subset\subset \mathbb{R}^m$, $p \in (1, \infty)$, then there exists $\theta > 0$ such that the following holds: Let $w \in L^2(\mathbb{R}^m) \cap L^{(2)^{2\mu}}(D)$, that is,*

$$\|w\|_{2, \mathbb{R}^m} + \sup_{B_\rho \subset D} \rho^{\frac{2\mu-m}{2}} \|w\|_{2, B_\rho} < \infty, \quad (1.7)$$

be a solution to (1.5), where Ω is of the form (1.6). If Ω satisfies moreover

$$\sup_{B_\rho(x), x \in D} \rho^{\frac{2\mu-m}{2}} \|A^l\|_{L^2} \leq \theta, \quad (1.8)$$

then $w \in L_{loc}^p(D)$.

Let us remark the following corollaries from Theorem 1.1.

As mentioned above, by the representation (1.4) this gives a new proof of Rivière's theorem [Riv07], and also the ε -regularity theorem of [RS08].

Moreover, from Theorem 1.1 a new proof of Sharp and Topping's integrability theorem [ST11] for (1.1) follows, and also an extension to the super-critical setting. The latter has been done independently, and by different methods by Sharp [Sha12].

Also, we extend these integrability results to the non-local case for $\mu \leq 1$. For $\mu > 1$ it seems already in the classical setting of the biharmonic maps, cf. [Str08], that for ε -regularity we need more information on the growth of Ω in terms of the solution, a fact which appeared also in our setting and forced us to restrict $\mu = \frac{m}{2}$ if $\mu > 1$.

Another corollary worth mentioning is that the arguments presented here also enable us to treat ε -regularity critical points of more general non-local energies, e.g.,

$$E(u) = \int |\nabla^\alpha u|^2 \quad u : \mathbb{R}^m \rightarrow \mathcal{N} \subset \mathbb{R}^N, \quad (1.9)$$

where for $\mathcal{R} = [\mathcal{R}_1, \dots, \mathcal{R}_m]^T$, and \mathcal{R}_i being the i -th Riesz transform,

$$\nabla^\alpha u := \mathcal{R}[|\nabla|^\alpha u].$$

Another remark regards the smallness condition of (1.8). In the critical setting $2\mu = m$, it is easy to verify, that this condition holds, if D is chosen appropriately small. In the super-critical regime $2\mu < m$, this condition would

follow from some kind of monotonicity formula for stationary points of energies of the form (1.9), which for the non-classical settings are unknown so far.

Let us now sketch the arguments we are going to need. Firstly, somewhat motivated by the arguments in [RS08], we are going to estimate the growth of the norm possibly far below the natural exponent 2. More precisely we estimate the growth in R of

$$\sup_{B_r \subset B_R} r^{\frac{\lambda_\kappa - m}{p_\kappa}} \|w\|_{p_\kappa, B_r}, \quad (1.10)$$

starting with $\kappa = \mu$, where

$$\lambda_\kappa := \frac{m(2\mu - \kappa)}{m - \kappa},$$

$$p_\kappa := \frac{m}{m - \kappa}.$$

The main work is to show that for any $\kappa \in [\mu, 2\mu)$ there is a good growth of these quantities, then starting for $\kappa_0 = \mu$, we can find a sequence of κ_i which converges to 2μ , such that each growth of the κ_i -norm (that is (1.10) with κ_i) is controlled by the κ_{i-1} -norm. Finally, for κ sufficiently close to 2μ , we show that we can actually have an estimate for $p > 2$. From this we have

Theorem 1.2. *There is $\theta_2 > 0$ such that if $\theta < \theta_2$, there exists $p > 2$, $\lambda < 2\mu$, such that*

$$w \in L_{loc}^{(p)\lambda}(D).$$

For Theorem 1.2, the antisymmetry of Ω will be crucial. Once Theorem 1.2 is established, the system (1.5) becomes subcritical, and we can drop the antisymmetry condition and just by the growth of the PDE, we have

Theorem 1.3. *Assume w as in Theorem 1.1, where we do not require the antisymmetry of Ω . Assume moreover, that $w \in L_{loc}^{p_1}(D)$ for $p_1 > 2$. Then for any $p > 2$, there is $\theta_p > 0$ such that if $\theta < \theta_p$ in (1.8), also*

$$w \in L_{loc}^p(D).$$

The main difficulty is thus Theorem 1.2 and the estimates of the Morrey norm. For the proof of this theorem we need the following two main technical ingredients: Firstly, we need to extend the known commutator results [DLR11b, DLR11a], and also the pointwise estimates [Sch11, Sch12]. We introduce the following commutators: Let X be a linear space, For $\varphi \in C_0^\infty(\mathbb{R}^m)$, $T : L^p(\mathbb{R}^m) \rightarrow L^q(\mathbb{R}^m)$, $1 \leq p, q \leq \infty$. We then set for $f \in L^p(\mathbb{R}^m)$ the commutator $\mathcal{C}(\varphi, T)[f]$

$$\mathcal{C}(\varphi, T)[f] := \varphi T[f] - T[\varphi f]. \quad (1.11)$$

This commutator was estimated in terms of Hardy spaces for $T = \mathcal{R}$ the Riesz transform or $T = I_s$ the Riesz potential in [CRW76, Cha82], nevertheless we need more precise estimates and generalizations. The next bilinear commutator was introduced in [DLR11b], in [Sch11] pointwise estimates were given.

$$H_s(a, b) := |\nabla|^s(ab) - a|\nabla|^s b - b|\nabla|^s a. \quad (1.12)$$

For these commutators we show the following

Theorem 1.4. *For any $\mu \in (0, 1]$, we have the following Hardy-space \mathcal{H} estimate (for \mathcal{R} any zero-multiplier operator, we need it for the Riesz-transform, only)*

$$\| |\nabla|^\mu (\mathcal{R}[h] I_\mu b - \mathcal{R}[h I_\mu b]) \|_{\mathcal{H}} \leq \|h\|_2 \|b\|_2$$

Moreover, we have

$$\|\mathcal{C}(f, \mathcal{R})[|\nabla|^\mu \varphi]\|_2 \lesssim \| |\nabla|^\mu f \|_2 [\varphi]_{BMO},$$

and its pointwise counter-part: For any $\delta_i \in (0, 1)$ and any $\gamma_i \in (0, \delta_i)$, $i = 1, 2$,

$$|\mathcal{C}(a, \mathcal{R})[b]| \leq C_{\mathcal{R}, \delta_1, \gamma_1} I_{\delta_1 - \gamma_1} \left| |\nabla|^{\delta_1} a \right| I_{\gamma_1} |b| + C_{\mathcal{R}, \delta_2, \gamma_2} I_{\gamma_2} \left(I_{\delta_2 - \gamma_2} |b| \left| |\nabla|^{\delta_2} a \right| \right).$$

Finally we have

$$\|H_\mu(\varphi, g)\|_2 \lesssim \| |\nabla|^\mu g \|_2 [\varphi]_{BMO}.$$

and

$$\|\nabla^\mu H_\mu(a, b)\|_{\mathcal{H}} \lesssim \|\nabla^\mu a\|_2 \|\nabla^\mu b\|_2 \quad \text{for } \mu \in (0, 1], \quad (1.13)$$

as well as its pointwise counterpart: for any $\mu \in [0, m]$ there is $L \in \mathbb{N}$ such that for any $\beta \in [0, \min(\mu, 1))$, $\mu \in [0, m)$, $\tau \in (\max\{\beta, \mu + \beta - 1\}, \mu]$ there are, $s_k \in [0, \mu)$, $t_k \in [0, \tau)$, where $\tau - \beta - s_k - t_k \geq 0$, such that the following holds

$$\left| |\nabla|^\beta H_\mu(a, b) \right| \lesssim \sum_{k=1}^L I_{\tau - \beta - s_k - t_k} (I_{s_k} |\nabla|^\mu a | I_{t_k} |\nabla|^\tau b).$$

Remark 1.5. For $\mu < 1$ the Hardy-space estimates above follow essentially from an obvious adaption of Da Lio and Rivière's argument [DLR11b], and (1.13) has been proven by them. For $\mu > 1$, already from the pointwise arguments in [Sch11] there is no hope for similar results. The interesting and new case $\mu = 1$, for which even (1.13) was unclear up to now, needs a more careful adaption of the arguments in [DLR11b].

Equipped with a good understanding of these commutator we will show

Theorem 1.6. There is a uniform $\Lambda > 0$ such that the following holds: Let Ω be as in (1.6), and assume that $\Omega_{ij}[\cdot] = -\Omega_{ji}[\cdot]$. For any $B_r \subset \mathbb{R}^m$, we can then choose $P : \mathbb{R}^m \rightarrow SO(N)$, $\text{supp}(P - I) \subset B_r$. Then for any $\varphi \in C_0^\infty(B_r)$,

$$-\int \Omega^P [|\nabla|^\mu \varphi] \leq C r^{\frac{m}{2} - \mu} \|A\|_2 [\varphi]_{BMO} + \|A\|_2^2 \begin{cases} [\varphi]_{BMO} & \text{if } \mu \in (0, 1], \\ \||\nabla|^\mu \varphi\|_{(2, \infty)} & \text{if } \mu > 1, \end{cases}$$

where

$$\Omega_{ij}^P[f] := (|\nabla|^\mu P_{ik}) P_{kj}^T f + P_{ik} \Omega_{kl} [P_{lj}^T f],$$

In [Sch10a] the construction of P is done via minimization of $E(P) = \|P \nabla P^T + P \Omega P\|_{L^2}^2$ under the condition that P maps into $SO(N)$, a.e.. This is the argument that Hélein [Hél91] essentially used for his moving-frame technique, and it provides an alternative to Rivière's adaption of Uhlenbecks [Uhl82] gauge-theoretic construction of P in [Riv07]. Both techniques can be extended to the fractional case, where Ω is still a pointwise multiplication [DLR11a, Sch11]. We adapt the arguments [Sch11, Sch10a] to this case of a non-local operator $\Omega[\cdot]$, by minimizing in Section 1.6 the energy

$$E(P) := \sup_{\psi \in L^2(\mathbb{R}^m)} \int \Omega^P[\psi],$$

and showing that several terms of the Euler-Lagrange equations fall under the realm of Theorem 1.4.

Notation Let $L^{p,q}$ be the Lorentz spaces, cf., e.g. [Hun66, Tar07, Gra08], whose norm we denote with $\|\cdot\|_{(p,q)}$. We set

$$\|f\|_{(p,q)_\lambda} \equiv \|f\|_{\mathcal{M}((p,q),\lambda)} := \sup_{B_r \subset \mathbb{R}^m} r^{\frac{\lambda-m}{p}} \|f\|_{(p,q), B_r}, \quad (1.14)$$

and for $A \subset \mathbb{R}^m$,

$$[f]_{(p,q)_\lambda, A} := |A|^{\frac{\lambda-m}{mp}} \|f\|_{(p,q), A}, \quad (1.15)$$

$$\|f\|_{(p,q)_\lambda, A} := \sup_{B_\rho \subset A} [f]_{(p,q), B_\rho}. \quad (1.16)$$

We say that f belongs to the Morrey space $L^{(p,q)_\lambda}(A)$, if the respective norm $\|f\|_{(p,q)_\lambda, A}$ is finite. We will also use frequently the following annuli

$$A_{\Lambda, r}^k := B_{2^k \Lambda r} \setminus B_{2^{k-1} \Lambda r}, \quad A_r^k \equiv A_{1, r}^k. \quad (1.17)$$

In Section A we recall several facts on the fractional laplacian, which we are going to use throughout this work.

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2. $L^{2+\varepsilon}$ -integrability: Proof of Theorem 1.2

It is helpful, to check once and for all,

$$m - 2\mu = \frac{m - \lambda_\kappa}{p_\kappa}, \quad \kappa \in [\mu, 2\mu), \quad (2.1)$$

where

$$\lambda_\kappa := \frac{m(2\mu - \kappa)}{m - \kappa}, \quad (2.2)$$

$$p_\kappa := \frac{m}{m - \kappa}. \quad (2.3)$$

Assume $w : \mathbb{R}^m \rightarrow \mathbb{R}^N$, $\mu \leq \frac{m}{2}$, $w \in L^2(\mathbb{R}^m)$, $|\nabla|^\mu w \in L^2(\mathbb{R}^m)$ is for $D \subset\subset \mathbb{R}^m$ a solution to (1.5). We are going to establish that for any $\kappa \in [\mu, 2\mu)$, if $\theta \equiv \theta_\kappa$ in (1.8) is suitably small, for any $\tilde{D} \subset\subset D$, we have

$$\sup_{r>0, x_0 \in \tilde{D}} r^{\frac{\lambda_\kappa - m}{p_\kappa}} \|w\|_{p_\kappa, B_r(x_0)} \leq C_{\tilde{D}, w, \kappa}. \quad (2.4)$$

Note that possibly $p_\kappa < 2$ for all $\kappa \in [\mu, 2\mu)$. In order to show (2.4), we first note that its satisfied by assumption (1.7) for $\kappa = \mu$. In fact, if $x_0 \in \tilde{D} \subset\subset D$, then for any $r > 0$, or $B_r(x_0) \subset D$ or $r > c \text{dist}(\tilde{D}, \partial D)$. Now, we show that for arbitrary $\kappa \in [\mu, 2\mu)$, there is $\kappa_1 > \kappa$, so that (2.4) holds. Moreover, we will show a lower bound on $\kappa_1 - \kappa$, in order to ensure that we come arbitrarily close to 2μ if we repeat this construction finitely many times. Then we can show that if we choose $\kappa \in [\mu, 2\mu)$ close enough to 2μ , (2.4) suffices to conclude the better integrability of Theorem 1.2.

Establishing (2.4)

For mappings $P : \mathbb{R}^m \rightarrow SO(N)$, $P \equiv I$ on $\mathbb{R}^m \setminus D$ (denoting with $I = (\delta_{ij})_{ij} \in \mathbb{R}^{N \times N}$ the identity matrix) from (1.5) we have

$$\begin{aligned} \int P_{ik} w_k |\nabla|^\mu \varphi &= \int w_k |\nabla|^\mu (P_{ik} \varphi) - \int w_k (|\nabla|^\mu P_{ik}) \varphi - \int w_k H_\mu(P_{ik}, \varphi) \\ &= - \int \Omega_{kl} [w_l] P_{ik} \varphi - \int w_k (|\nabla|^\mu P_{ik}) \varphi - \int w_k H_\mu((P - I)_{ik}, \varphi) \end{aligned}$$

Setting $v_i := P_{ik} w_k$, this is

$$\int v_i |\nabla|^\mu \varphi = - \int (P_{ik} \Omega_{kl} [P_{jl} v_j] + (|\nabla|^\mu P_{ik}) P_{jk} v_j) \varphi - \int w_k H_\mu((P - I)_{ik}, \varphi) \quad (2.5)$$

The Growth Estimates. From (2.5), Lemma 2.2, and Lemma 2.3 we infer

Theorem 2.1 (Right-hand side estimates). *If $\mu \in (0, \min\{1, \frac{m}{2}\}]$ or $2\mu = m$, there is a uniform $\Lambda \equiv \Lambda_\mu > 0$, depending only on μ , such that the following holds: Let $B_r \subset \mathbb{R}^m$, and assume (2.5) holds for all $\varphi \in C_0^\infty(B_r)$. Then exists a choice of P such that (2.5) implies for any $\varphi \in C_0^\infty(B_{\Lambda^{-1}r})$, and for any $\tau \in (0, \mu]$ sufficiently close to, or greater than $2\mu - \kappa$,*

$$\begin{aligned} (\Lambda^{-1}r)^{2\mu - m} \int v |\nabla|^\mu \varphi &\leq C_\kappa \theta \| |\nabla|^\tau \varphi \|_{(\frac{m}{\tau + \kappa - \mu}, 1)} \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_r} \\ &\quad + C_\kappa \theta \| |\nabla|^\tau \varphi \|_{(\frac{m}{\tau + \kappa - \mu}, 2)} \Lambda^{\kappa - 3\mu} \sum_{k=1}^{\infty} 2^{k(\kappa - 3\mu)} [w]_{(p_\kappa, \infty)_{\lambda_\kappa}, A_r^k}. \end{aligned}$$

where we recall that the right-hand side norms were defined in (1.15), (1.16), A_r^k is as in (1.17), and λ_κ as in (2.2), p_κ as in (2.3).

From Theorem 2.1 and Lemma C.1 (applied to $\Lambda^{-1}r$ instead of r) we infer for any $\tau \in (0, \mu]$ sufficiently close to μ and any $\Lambda \gg \Lambda_\mu$ sufficiently large (for the right-hand side norms recall (1.15) and (1.16)), also in view of Proposition A.7,

$$\begin{aligned}
& (\Lambda^{-2}r)^{2\mu-m} \|\ |\nabla|^{\mu-\tau} v \|_{(\frac{m}{m+\mu-\tau-\kappa}, \infty), B_{\Lambda^{-2}r}} \\
\leq & \Lambda^{-1} C_\kappa \theta \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_r} \\
& + \Lambda^{-1} C_\kappa \theta \sum_{k=1}^{\infty} 2^{k(\kappa-3\mu)} [w]_{(p_\kappa, \infty)_{\lambda_\kappa}, A_r^k} \\
& + C (\Lambda^{-2}r)^{2\mu-m} \Lambda^{\kappa-m+\tau-\mu} \|w\|_{(p_\kappa, \infty), B_{\Lambda^{-1}r}} \\
& + C (\Lambda^{-2}r)^{2\mu-m} \Lambda^{\kappa-m+\tau-\mu} \sum_{k=0}^{\infty} 2^{k(\kappa-m+\tau-\mu)} \|w\|_{(p_\kappa, \infty), A_{\Lambda^{-1}r}^k} \\
\stackrel{(2.1)}{\leq} & C_\kappa \theta \Lambda^{m-2\mu} \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_r} \\
& + C_\kappa \theta \Lambda^{m-2\mu} \sum_{k=1}^{\infty} 2^{k(\kappa-3\mu)} [w]_{(p_\kappa, \infty)_{\lambda_\kappa}, A_r^k} \\
& + C \Lambda^{\kappa+\tau-3\mu} \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_{\Lambda^{-1}r}} \\
& + C \Lambda^{\kappa+\tau-3\mu} \sum_{k=0}^{\infty} 2^{k(\kappa+\tau-3\mu)} [w]_{(p_\kappa, \infty)_{\lambda_\kappa}, A_{\Lambda^{-1}r}^k} \\
\stackrel{P.A.7}{\lesssim} & (C_\kappa \theta \Lambda^{m-2\mu} + C \Lambda^{\kappa+\tau-3\mu}) \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_r} \\
& + (C_\kappa \theta \Lambda^{m-2\mu} + C \Lambda^{\kappa+\tau-3\mu}) \sum_{k=1}^{\infty} 2^{k(\kappa+\tau-3\mu)} [w]_{(p_\kappa, \infty)_{\lambda_\kappa}, A_r^k}.
\end{aligned}$$

For later reference, we write this as

$$\begin{aligned}
(\Lambda^{-2}r)^{2\mu-m} \|\ |\nabla|^{\mu-\tau} v \|_{(\frac{m}{m+\mu-\tau-\kappa}, \infty), B_{\Lambda^{-2}r}} & \leq (C_{\kappa, \mu} \theta \Lambda^{m-2\mu} + C_\mu \Lambda^{\kappa+\tau-3\mu}) \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_r} \\
& + (C_{\kappa, \mu} \theta \Lambda^{m-2\mu} + C_\mu \Lambda^{\kappa+\tau-3\mu}) \sum_{k=1}^{\infty} 2^{k(\kappa+\tau-3\mu)} [w]_{(p_\kappa, \infty)_{\lambda_\kappa}, A_r^k}.
\end{aligned} \tag{2.6}$$

For $\tau = \mu$,

$$\begin{aligned}
(\Lambda^{-2}r)^{2\mu-m} \|v\|_{(p_\kappa, \infty), B_{\Lambda^{-2}r}} & \leq (C_{\kappa, \mu} \theta \Lambda^{m-2\mu} + C_\mu \Lambda^{\kappa-2\mu}) \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_r} \\
& + (C_{\kappa, \mu} \theta \Lambda^{m-2\mu} + C_\mu \Lambda^{\kappa-2\mu}) \sum_{k=1}^{\infty} 2^{k(\kappa-2\mu)} [w]_{(p_\kappa, \infty)_{\lambda_\kappa}, A_r^k}.
\end{aligned} \tag{2.7}$$

The Iteration Procedure. Note that $|w| = |v|$, so we can use them equivalently. Equation (2.7) holds for any $B_r(x_0)$, where $x_0 \in D$ and $r < \tilde{d}(x_0) := C \operatorname{dist}(x_0, \partial D)$ (the constant essentially only depending on the construction of P and the set where Ω is small). For $x_0 \in D$ and $R > 0$ set

$$\Phi_{x_0}(R) := \sup_{B_\rho \subset B_R(x_0)} \rho^{2\mu-m} \|w\|_{(p_\kappa, \infty), B_\rho},$$

and its centered counter-part

$$\Psi_{x_0}(R) := \sup_{\rho \in (0, R)} \rho^{2\mu-m} \|w\|_{(p_\kappa, \infty), B_\rho(x_0)} \leq \Phi_{x_0}(R)$$

then from (2.7) for any $R, x_0 \in D$ with $R < d(x_0)$, we have

$$\Phi_{x_0}(\Lambda^{-1}R) \leq (C_\kappa \theta \Lambda^{m-2\mu} + C \Lambda^{\kappa-2\mu}) \Phi_{x_0}(R) + (C_\kappa \theta \Lambda^{m-2\mu} + C \Lambda^{\kappa-2\mu}) \sum_{k=1}^{\infty} 2^{k(\kappa-2\mu)} \Psi_{x_0}(2^k R).$$

Note that from (2.4), we know that $\Phi_{x_0}(R) < C_{D, x_0, w}$ for any $x_0 \in D$. Now we can iterate, Lemma D.1, satisfying the assumption (D.1) by choosing $\Lambda \equiv \Lambda_\kappa := 2 \frac{C_\mu}{(2\mu - \kappa)^4}$ with sufficiently large C_μ and assuming that $\theta < (\Lambda_\kappa)^{\kappa - m}$. Then, for any $r < R$,

$$\sup_{B_\rho \subset B_r(x_0)} \rho^{2\mu - m} \|w\|_{(p_\kappa, \infty), B_\rho} = \sup_{B_\rho \subset B_r(x_0)} \rho^{2\mu - m} \|v\|_{(p_\kappa, \infty), B_\rho} \lesssim C_{\kappa, w, \Lambda, R} r^{\sigma_\kappa}, \quad \text{where } \sigma_\kappa = \frac{(2\mu - \kappa)^4}{C_\mu}.$$

We can assume, that $\sigma_\kappa < 2\mu - \kappa$. Since

$$\sup_{\rho > R} r^{2\mu - \sigma_\kappa - m} \|w\|_{(p_\kappa, \infty), B_r(x_0)} \lesssim R^{-\sigma_\kappa} \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_{4r}},$$

we arrive at

$$r^{2\mu - \sigma_\kappa - m} \|w\|_{(p_\kappa, \infty), B_r(x_0)} \leq C_{\kappa, w, x_0}$$

so we get for any $B_r(x_0) \subset B_R(x_0)$

$$r^{-\sigma_\kappa} \|\chi_{B_r} w\|_{(p_\kappa, \infty)_{\lambda_\kappa}} + \sup_{\rho > 0} \rho^{2\mu - \sigma_\kappa - m} \|w\|_{(p_\kappa, \infty), B_\rho(x_0)} \leq C_{\kappa, w}.$$

Plugging this into (2.6), we have for all $\tau \in (0, \mu]$ sufficiently close to, or greater than $2\mu - \kappa$,

$$r^{2\mu - m} \|\nabla^{|\mu - \tau|} v\|_{(\frac{m}{m + \mu - \tau - \kappa}, \infty), B_{\lambda^{-1} r}(x_0)} \lesssim C_{\kappa, w} r^{\sigma_\kappa} + C_{\kappa, w} r^{\sigma_\kappa} \sum_{k=1}^{\infty} 2^{k(\kappa - 2\mu + \sigma_\kappa)}, \quad (2.8)$$

so that we have for all small r ,

$$r^{2\mu - m - \sigma_\kappa} \|\nabla^{|\mu - \tau|} v\|_{(\frac{m}{m + \mu - \tau - \kappa}, \infty), B_r(x_0)} \lesssim C_{w, \kappa, R}.$$

Moving the $B_R(x_0)$, for any $D_1 \subset\subset D$, we have that

$$\|\nabla^{|\mu - \tau|} v\|_{(\frac{m}{m + \mu - \tau - \kappa}, \infty)_{\lambda}, D_1} \lesssim C_{w, \kappa, |\nabla|_1, D} \quad (2.9)$$

for λ such that (choosing τ possibly even closer to μ , ensuring that $|\mu - \tau| \leq \frac{\sigma_\kappa}{2}$)

$$\frac{\lambda}{m} = \frac{3\mu - \tau - \kappa - \sigma_\kappa}{m + \mu - \tau - \kappa} \leq \frac{3\mu - \tau - \kappa - \sigma_\kappa}{m - \kappa} \stackrel{(2.2)}{=} \frac{\lambda_\kappa}{m} + \frac{\mu - \tau - \sigma_\kappa}{m - \kappa}$$

Choosing the next κ . Assume for a moment that $2\mu < m$. we can guarantee

$$0 < \lambda < \lambda_\kappa - c_m \sigma_\kappa, \quad (2.10)$$

and we choose $\kappa_{1,1} \in (\kappa, 2\mu)$ via

$$\lambda =: m \frac{2\mu - \kappa_{1,1}}{m - \kappa_{1,1}}.$$

By (2.10),

$$m \frac{2\mu - \kappa_{1,1}}{m - \kappa_{1,1}} < m \frac{2\mu - \kappa}{m - \kappa} - c_{m-2\mu} \sigma_\kappa$$

and thus we have

$$\kappa_{1,1} > \kappa + \sigma c_{m-2\mu} \frac{(m - \kappa_{1,1})(m - \kappa)}{m}.$$

On the other hand, by a localized version of Adams' [Ada75]-argument on Riesz potentials, we infer from (2.9) that for any $D_2 \subset\subset D_1$,

$$\|v\|_{(p, \infty)_{\lambda}, D_2} = \|w\|_{(p, \infty)_{\lambda}, D_2} < \infty,$$

where

$$\frac{1}{p} = \frac{m + \mu - \tau - \kappa}{m} - \frac{\mu - \tau}{\lambda} \in (0, 1).$$

Letting

$$\frac{m}{m - \kappa_{1,2}} := p,$$

we can estimate

$$\frac{\kappa_{1,2} - \kappa}{m} = (\mu - \tau) \left(\frac{1}{\lambda} - \frac{1}{m} \right) \geq \sigma_\kappa c_\mu.$$

Thus for a certain $\alpha > 0$,

$$\kappa_1 := \min \kappa_{1,1}, \kappa_{1,2} \geq \kappa_0 + c_0(2\mu - \kappa)^\alpha,$$

and since

$$p \geq \frac{m}{m - \kappa_1}, \quad \lambda < m \frac{2\mu - \kappa_1}{m - \kappa_1},$$

for any $D_3 \subset\subset D$, we arrive at

$$\|w\|_{(p_{\kappa_1}, \infty)_{m \frac{2\mu - \kappa_1}{m - \kappa_1}}, D_3} < \infty.$$

Varying this in $D_3 \subset\subset D$, we have (2.4) for κ_1 . If $2\mu = m$, we use this same argument, to conclude that $w \in L^p(D_3)$ for some $p > 2$, which is already the claim of Theorem 1.2.

Estimating the growth of κ . Iterating this procedure (for smaller and smaller θ in (1.8)), we obtain $\kappa_k \in [\mu, 2\mu)$, and

$$\kappa_{k+1} \geq \kappa_k + c_0(2\mu - \kappa)^\alpha.$$

Since the sequence $(\kappa_k)_k$ is monotone and bounded, and the only fixed point is $\kappa_\infty = 2\mu$, for any $\varepsilon > 0$ there is a step-count L such that $|\kappa_L - 2\mu| < \varepsilon$. This shows (2.4).

Integrability slightly above 2

By the arguments above, fixing $\tilde{D} \subset\subset D$, going back to (2.8), if $2\mu - \kappa < \varepsilon$ small enough, for $\tau \in (\varepsilon, \mu]$, ignoring $\sigma_\kappa > 0$,

$$\sup_{B_r \subset \tilde{D}} r^{2\mu - m} \|\ |\nabla|^{\mu - \tau} v \|_{(\frac{m}{m + \mu - \tau - \kappa}, \infty), B_{\Lambda^{-1}\tau}(x_0)} \lesssim C_{\kappa, w, \tilde{D}}.$$

If $2\mu = m$, choosing $\tau = \mu$, we have

$$\frac{m}{m + \mu - \mu - \kappa} \xrightarrow{\kappa \rightarrow 2\mu = m} \infty,$$

which proves Theorem 1.2, and in fact even Theorem 1.1. So let from now on $2\mu < m$, $\mu \leq 1$. Then for $\lambda_{s,\varepsilon} \in (0, m)$, $s := \mu - \tau$,

$$\begin{aligned} \frac{\lambda_{s,\varepsilon} - m}{\frac{m}{m + \mu - \tau - \kappa}} &= 2\mu - m \\ \Leftrightarrow \lambda_{s,\varepsilon} &= \frac{m}{m + \mu - \tau - \kappa} (3\mu - \tau - \kappa) \xrightarrow{\tau \rightarrow \mu, \kappa \rightarrow 2\mu} 0 \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\tilde{p}} &:= \frac{m + \mu - \tau - \kappa}{m} - \frac{\mu - \tau}{\frac{m}{m + \mu - \tau - \kappa} (3\mu - \tau - \kappa)} = \frac{m + \mu - \tau - \kappa}{m} - \frac{(\mu - \tau)(m + \mu - \tau - \kappa)}{m(3\mu - \tau - \kappa)} \\ &= 1 + \frac{\mu - \tau - \kappa}{m(3\mu - \tau - \kappa)} (2\mu + \tau - \kappa) - \frac{(\mu - \tau)}{3\mu - \tau - \kappa} \end{aligned}$$

we have by Adams' [Ada75],

$$v \in L_{loc}^{(\tilde{p}, \infty)\lambda_{s,\varepsilon}}(D).$$

One checks that there is $\varepsilon > 0$ such that if $|\tau - \mu| < \varepsilon$, $2\kappa - \mu < \varepsilon$, then $\tilde{p} > 2$, $\lambda_{s,\varepsilon} < 2\mu$.

2.1. Estimates of the H -term: Proof of Theorem 2.1 (I)

This is to estimate for $\varphi \in C_0^\infty(B_r)$ the following term

$$\int w H_\mu(P - I, \varphi) = \int I_\beta w |\nabla|^\beta H_\mu(P, \varphi) \quad (2.11)$$

Lemma 2.2. *Let $\mu \in (0, \frac{m}{2}]$, $\mu \leq 1$ or $\mu = \frac{m}{2}$. For any $\kappa \in [\mu, 2\mu)$, there are $C_{\kappa, \mu} > 0$, $\tau \in (0, \mu)$ such for any $\varphi \in C_0^\infty(B_{\Lambda^{-1}r})$ the following holds: If $\text{supp}(P - I) \subset B_{\Lambda^{-1}r}$,*

$$\begin{aligned} (\Lambda^{-1}r)^{2\mu-m} \int w H_\mu(P - I, \varphi) &\leq C_{\kappa, \mu} \|\nabla|^\tau \varphi\|_{(\frac{m}{\kappa+\tau-\mu}, 2)} (\Lambda^{-1}r)^{\mu-\frac{m}{2}} \|\nabla|^\mu P\|_2 \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_r} \\ &\quad + C_{\kappa, \mu} \|\nabla|^\tau \varphi\|_{(\frac{m}{\kappa+\tau-\mu}, 2)} (\Lambda^{-1}r)^{\mu-\frac{m}{2}} \|\nabla|^\mu P\|_2 \sum_{k=1}^{\infty} (2^k \Lambda)^{\kappa-3\mu} [w]_{(p_\kappa, \infty)_{\lambda_\kappa}, A_r^k} \end{aligned}$$

where we recall the definition A_r^k from (1.17), λ_κ from (2.2), and p_κ from (2.3). As for the asymptotic behavior as $\kappa \rightarrow 2\mu$, one can choose τ approaching $\max\{\mu - 1, 0\}$, and $C_{\kappa, \mu}$ blows up.

Proof of Lemma 2.2. For a somewhat clearer presentation, we are going to show the following claim for $\varphi \in C_0^\infty(B_r)$ and $\text{supp}(P - I) \subset B_r$

$$\begin{aligned} r^{2\mu-m} \int w H_\mu(P - I, \varphi) &\leq C_{\kappa, \mu} \|\nabla|^\tau \varphi\|_{(\frac{m}{\kappa+\tau-\mu}, 2)} r^{\mu-\frac{m}{2}} \|\nabla|^\mu P\|_2 \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_{\Lambda r}} \\ &\quad + C_{\kappa, \mu} \|\nabla|^\tau \varphi\|_{(\frac{m}{\kappa+\tau-\mu}, 2)} r^{\mu-\frac{m}{2}} \|\nabla|^\mu P\|_2 \sum_{k=1}^{\infty} (2^k \Lambda)^{\kappa-3\mu} [w]_{(p_\kappa, \infty)_{\lambda_\kappa}, A_{\Lambda, r}^k}. \end{aligned}$$

Applied to $\tilde{r} := \Lambda^{-1}r$ gives the original claim.

As usual, we decompose

$$\int w H_\mu(P - I, \varphi) = I + \sum_{k=1}^{\infty} II_k,$$

where

$$I := \int \chi_{B_{\Lambda r}} w H_\mu(P - I, \varphi),$$

and, denoting $A_k := A_{\Lambda, r}^k$,

$$II_k := \int w H_\mu(P - I, \varphi) \chi_{A_k}.$$

As for II_k , since $\text{supp} \varphi \cup \text{supp}(P - I) \subset \overline{B_r}$

$$H_\mu(P - I, \varphi) \chi_{A_k} = \chi_{A_k} |\nabla|^\mu ((P - I)\varphi).$$

By Lemma B.1 we then have for any $\tau \in (0, \mu]$, using also Lemma A.5,

$$\begin{aligned} \|H_\mu(P - I, \varphi)\|_{(\frac{m}{\kappa}, 1)_{A_k}} &\lesssim (2^k \Lambda r)^{-m-\mu} (2^k \Lambda r)^\kappa r^{\frac{m}{2}-\kappa+\mu} \|\varphi\|_{(\frac{m}{\kappa-\mu}, \infty)} \|P - I\|_2 \\ &\lesssim (2^k \Lambda r)^{-m-\mu} (2^k \Lambda r)^\kappa r^{\frac{m}{2}-\kappa+2\mu} \|\nabla|^\tau \varphi\|_{(\frac{m}{\kappa+\tau-\mu}, \infty)} \|\nabla|^\mu P\|_2 \\ &= (2^k \Lambda)^{-m+\kappa-\mu} \|\nabla|^\tau \varphi\|_{(\frac{m}{\kappa+\tau-\mu}, \infty)} r^{\mu-\frac{m}{2}} \|\nabla|^\mu P\|_2. \end{aligned}$$

Consequently,

$$\begin{aligned} |II_k| &\lesssim \|w \chi_{A_k}\|_{(p_\kappa, \infty)} (2^k \Lambda)^{-m+\kappa-\mu} \|\nabla|^\tau \varphi\|_{(\frac{m}{\kappa+\tau-\mu}, \infty)} r^{\mu-\frac{m}{2}} \|\nabla|^\mu P\|_2 \\ &\stackrel{(2.1)}{\lesssim} (2^k \Lambda r)^{m-2\mu} [w \chi_{A_k}]_{(p_\kappa, \infty)_{\lambda_\kappa}} (2^k \Lambda)^{-m+\kappa-\mu} \|\nabla|^\tau \varphi\|_{(\frac{m}{\kappa+\tau-\mu}, \infty)} r^{\mu-\frac{m}{2}} \|\nabla|^\mu P\|_2 \\ &\lesssim r^{m-2\mu} (2^k \Lambda)^{\kappa-3\mu} \|\nabla|^\tau \varphi\|_{(\frac{m}{\kappa+\tau-\mu}, \infty)} r^{\mu-\frac{m}{2}} \|\nabla|^\mu P\|_2 [w \chi_{A_k}]_{(p_\kappa, \infty)_{\lambda_\kappa}}. \end{aligned}$$

As for I , set $\tilde{w} := \chi_{B_{\lambda r}} w$ and write

$$\int \tilde{w} H_\mu(P, \varphi) = \int I_\beta \tilde{w} |\nabla|^\beta H_\mu(P, \varphi)$$

Actually, the claim follows quite straight forward from (4.18) for $\mu \leq 1$, $\beta := \mu$, but the pointwise estimates on H , Lemma 4.1, are strong enough to deal with our situation, and they do not make use of para-products which were necessary for the proof of (4.18): By Lemma A.6

$$\|I_\beta \tilde{w}\|_{(p_1, \infty)_{\lambda \kappa}} \lesssim \|\tilde{w}\|_{(p_\kappa, \infty)_{\lambda \kappa}}$$

where for $\beta < \min(2\mu - \kappa, 1)$,

$$\frac{1}{p_1} = \frac{m - \kappa}{m} \frac{2\mu - \kappa - \beta}{2\mu - \kappa} \in (0, 1).$$

If $\mu = \frac{m}{2}$, we set $\beta = 0$, if $\mu < \frac{m}{2}$, let $\epsilon > 0$ such that $\mu + \epsilon < \frac{m}{2}$. Now we estimate $\|\nabla|^\beta H_\mu(P, \varphi)\|$, applying Lemma 4.1 for any $\tau \in (\max\{\beta, \mu + \beta - 1\}, \mu]$, we have to control terms of the form (for $s \in (0, \mu)$, $t \in (0, \tau)$, $\tau - \beta - s - t \in [0, \epsilon)$)

$$I_{\tau - \beta - s - t}(I_s |\nabla|^\mu P | I_t |\nabla|^\tau \varphi).$$

We have

$$\begin{aligned} \|I_s |\nabla|^\mu P\|_{(p_2, 2)} &\lesssim \| |\nabla|^\mu P \|_2, \quad \frac{1}{p_2} = \frac{1}{2} - \frac{s}{m} \in (0, 1), \\ \|I_t |\nabla|^\tau \varphi\|_{(p_3, 2)} &\lesssim \| |\nabla|^\tau \varphi \|_{(\frac{m}{\kappa + \tau - \mu}, 2)}, \quad \frac{1}{p_3} = \frac{\kappa + \tau - \mu}{m} - \frac{t}{m} \in (0, 1). \end{aligned}$$

Note that

$$0 < \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{2} + \frac{\kappa + \tau - \mu - s - t}{m} < \frac{1}{2} + \frac{\kappa + \tau - \mu + \epsilon - \tau + \beta}{m} < \frac{1}{2} + \frac{\epsilon + \mu}{m} < 1,$$

consequently,

$$\|I_{\tau - \beta - s - t}(I_s |\nabla|^\mu P | I_t |\nabla|^\tau \varphi)\|_{(p_4, 1)} \lesssim \| |\nabla|^\mu P \|_2 \| |\nabla|^\tau \varphi \|_{(\frac{m}{\kappa + \tau - \mu}, 2)},$$

where

$$\frac{1}{p_4} = \frac{1}{p_2} + \frac{1}{p_3} - \frac{\tau - \beta - s - t}{m} = \frac{1}{2} + \frac{\kappa + \beta - \mu}{m} \in (0, 1).$$

Now we have to ensure that the $f(\beta) \leq 1$ for admissible β (and admissible τ):

$$f(\beta) := \frac{1}{p_1} + \frac{1}{p_4} = \frac{3}{2} - \frac{\mu}{m} - \beta \frac{m - 2\mu}{m(2\mu - \kappa)} > 0.$$

Obviously, $f(0) = 1$ holds, if $\mu = \frac{m}{2}$ (so $\beta = 0$, and τ arbitrarily between $(\mu - 1, \mu]$). As for the case $\mu < \frac{m}{2}$, $\mu \leq 1$, We have $2\mu - \kappa \leq 1$ for $\kappa \in [\mu, 2\mu)$, then

$$f(2\mu - \kappa) = \frac{1}{2} + \frac{\mu}{m} < 1.$$

so we can take $\beta < 1$ sufficiently close to $2\mu - \kappa$, so that $f(\beta) < 1$, and take $\tau \in (\beta, \mu)$ sufficiently close to or greater than $2\mu - \kappa$. Consequently,

$$\begin{aligned} |I| &\lesssim \int_{B_{4r}} I_\beta \tilde{w} |\nabla|^\beta H_\mu(P, \varphi) + \sum_{k=1}^{\infty} \int_{A_{4r}^k} I_\beta \tilde{w} |\nabla|^\beta H_\mu(P, \varphi) \\ &\lesssim \|I_\beta \tilde{w}\|_{(p_1, \infty), B_{4r}} \| |\nabla|^\beta H_\mu(P, \varphi) \|_{(p_4, 1)} r^{m - \frac{m}{p_1} - \frac{m}{p_4}} \\ &\quad + \sum_{k=1}^{\infty} \|I_\beta \tilde{w}\|_{(p_1, \infty), A_{4r}^k} \| |\nabla|^\beta H_\mu(P, \varphi) \|_{(p_4, 1), A_{4r}^k} (2^k r)^{m - \frac{m}{p_1} - \frac{m}{p_4}} \\ &\lesssim r^{\frac{m - \lambda \kappa}{p_1}} \|I_\beta \tilde{w}\|_{(p_1, \infty)_{\lambda \kappa}} \| |\nabla|^\beta H_\mu(P, \varphi) \|_{(p_4, 1)} r^{m - \frac{m}{p_1} - \frac{m}{p_4}} \\ &\quad + \sum_{k=1}^{\infty} (2^k r)^{\frac{m - \lambda \kappa}{p_1}} \|I_\beta \tilde{w}\|_{(p_1, \infty)_{\lambda \kappa}} \| |\nabla|^\beta H_\mu(P, \varphi) \|_{(p_4, 1), A_{4r}^k} (2^k r)^{m - \frac{m}{p_1} - \frac{m}{p_4}} \\ &\lesssim r^{\frac{m - \lambda \kappa}{p_1}} \|\tilde{w}\|_{(p_\kappa, \infty)_{\lambda \kappa}} \| |\nabla|^\beta H_\mu(P, \varphi) \|_{(p_4, 1)} r^{m - \frac{m}{p_1} - \frac{m}{p_4}} \\ &\quad + \sum_{k=1}^{\infty} (2^k r)^{\frac{m - \lambda \kappa}{p_1}} \|\tilde{w}\|_{(p_\kappa, \infty)_{\lambda \kappa}, A_{4r}^k} \| |\nabla|^\beta H_\mu(P, \varphi) \|_{(p_4, 1), A_{4r}^k} (2^k r)^{m - \frac{m}{p_1} - \frac{m}{p_4}}. \end{aligned}$$

By Proposition 4.3, for the same τ as above,

$$\| |\nabla|^\beta H_\mu(P, \varphi) \|_{(p_4, 1)} \lesssim \| |\nabla|^\mu P \|_2 \| |\nabla|^\tau \varphi \|_{\frac{m}{\kappa - \tau - \mu}, 2}.$$

Now we apply Proposition 4.4 (using that φ and $P - I$ have support in B_r), and using

$$\begin{aligned} & \frac{m - \frac{m(2\mu - \kappa)}{m - \kappa}}{p_1} + m - \frac{m}{p_1} - \frac{m}{p_4} - m - \beta + \frac{m}{p_4} \\ &= -2\mu + \kappa, \end{aligned}$$

and

$$\frac{m - \lambda_\kappa}{p_1} + m - \frac{m}{p_1} - \frac{m}{p_4} + \frac{m}{2} - \mu = m - 2\mu$$

we conclude

$$\begin{aligned} |I| &\lesssim r^{m-2\mu} \| \tilde{w} \|_{(p_\kappa, \infty)_{\lambda_\kappa}} r^{\mu - \frac{m}{2}} \| |\nabla|^\mu P \|_2 \| |\nabla|^\tau \varphi \|_{\frac{m}{\kappa - \tau - \mu}, 2} \\ &\quad + r^{m-2\mu} \sum_{k=1}^{\infty} 2^{k(-2\mu + \kappa)} \| \tilde{w} \|_{(p_\kappa, \infty)_{\lambda_\kappa}, A_{4r}^k} \| |\nabla|^\tau \varphi \|_{(\frac{m}{\kappa + \tau - \mu}, \infty)} r^{\mu - \frac{m}{2}} \| |\nabla|^\mu P \|_{(2, \infty)} \\ &\lesssim C_\kappa r^{m-2\mu} \| w \chi_{B_{\Lambda r}} \|_{(p_\kappa, \infty)_{\lambda_\kappa}} r^{\mu - \frac{m}{2}} \| |\nabla|^\mu P \|_2 \| |\nabla|^\tau \varphi \|_{\frac{m}{\kappa - \tau - \mu}, 2}. \end{aligned}$$

□

2.2. Better integrability for transformed potential: Proof of Theorem 2.1 (II)

This section is devoted to the proof of the following Lemma:

Lemma 2.3. *Let $B_r \subset \mathbb{R}^m$, Ω as in (1.6), $\Lambda > 2$. There exists $P : \mathbb{R}^m \rightarrow SO(N)$, $P \equiv I$ on $\mathbb{R}^m \setminus B_{\Lambda^{-1}r}$, with the estimate*

$$(\Lambda^{-1}r)^{\frac{2\mu-m}{2}} \|\nabla|\mu P\|_{2,\mathbb{R}^m} \lesssim \theta, \quad (2.12)$$

such that for any $\tau \in (0, \mu]$ sufficiently close or greater than $2\mu - \kappa$, $\kappa \in [\mu, 2\mu)$, $\theta > 0$ from (1.8) in $D = B_r$, and for any $\varphi \in C_0^\infty(B_{\Lambda^{-1}r})$, if $\mu \in (0, 1]$, or $\mu = \frac{m}{2}$,

$$\begin{aligned} (\Lambda^{-1}r)^{2\mu-m} \int ((|\nabla|\mu P)P^T w + P\Omega[P^T w]) \varphi &\leq C_{\kappa,\mu} \theta \|\nabla|\tau \varphi\|_{(\frac{m}{\kappa+\tau-\mu}, 1)} \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_r} \\ &\quad + C_{\kappa,\mu} \theta \|\nabla|\tau \varphi\|_{(\frac{m}{\kappa+\tau-\mu}, 2)} \sum_{k=1}^{\infty} (2^k \Lambda)^{\kappa-3\mu} [w]_{(p_\kappa, \infty)_{\lambda_\kappa}, A_r^k}. \end{aligned}$$

where we recall the definition A_r^k from (1.17), λ_κ from (2.2), and p_κ from (2.3).

As in the proof of Lemma 2.2, we prove the scaled claim for replacing r by Λr which makes the presentation of the proof somewhat lighter: We are going to show the existence of P such that for $\varphi \in C_0^\infty(B_r)$

$$\begin{aligned} r^{2\mu-m} \int ((|\nabla|\mu P)P^T w + P\Omega[P^T w]) \varphi &\leq C_{\kappa,\mu} \theta \|\nabla|\tau \varphi\|_{(\frac{m}{\kappa+\tau-\mu}, 1)} \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_{\Lambda r}} \\ &\quad + C_{\kappa,\mu} \theta \|\nabla|\tau \varphi\|_{(\frac{m}{\kappa+\tau-\mu}, 2)} \sum_{k=1}^{\infty} (2^k \Lambda)^{\kappa-3\mu} [w]_{(p_\kappa, \infty)_{\lambda_\kappa}, A_{\Lambda r}^k}, \end{aligned} \quad (2.13)$$

Fix $B_r \subset \mathbb{R}^m$. In order to prove this claim, note that

$$\int ((|\nabla|\mu P)P^T w + P\Omega[P^T w]) \varphi = \int ((|\nabla|\mu P)P^T w + P\chi_{B_r}\Omega[P^T w]) \varphi,$$

so we are going to assume that the A_l in (1.6)

$$\text{supp } A_l \subset B_r, \quad \Omega[] = \chi_{B_r}\Omega[] \quad (2.14)$$

and consequently assuming (from (1.8)) that

$$r^{\frac{2\mu-m}{2}} \|A_l\|_{2,\mathbb{R}^m} \|f\|_2 + \sup_{\rho \in (0, \Lambda r)} \rho^{\frac{2\mu-m}{2}} \|\Omega[f]\|_{1, B_\rho} \lesssim \theta \|f\|_2 \quad (2.15)$$

Let $P : \mathbb{R}^m \rightarrow SO(N)$ be the minimizer, $P \equiv I$ on $\mathbb{R}^m \setminus B_r$, of $E(\cdot) \equiv E_{r,x,\Lambda_\mu,1,2}(\cdot)$, where Λ_μ is from Lemma 5.5. Using (5.7), (2.14), we have the estimates (for from now on fixed $\Lambda > 2$),

$$r^{\frac{2\mu-m}{2}} \|\nabla|\mu P\|_{2,\mathbb{R}^m} \lesssim \theta, \quad (2.16)$$

which after rescaling amounts to (2.12), and with the help of (2.15),

$$r^{\frac{2\mu-m}{2}} \|(|\nabla|\mu P)P^T f + P\Omega[P^T f]\|_{1, B_{\Lambda r}} \lesssim \theta \|f\|_{2,\mathbb{R}^m}. \quad (2.17)$$

Let

$$w = w\chi_{B_{\Lambda r}} + \sum_{k=1}^{\infty} w\chi_{A_{\Lambda r}^k} =: w_0 + \sum_{k=1}^{\infty} w_k.$$

Then,

$$\begin{aligned} &\int ((|\nabla|\mu P)P^T w + P\Omega[P^T w]) \varphi \\ &= \int (|\nabla|\mu P)P^T w_0 \varphi + P\Omega[P^T w_0 \varphi] - \int P\mathcal{C}(\varphi, \Omega)[P^T w_0] + \sum_{k=1}^{\infty} \int P\Omega[P^T w_k] \varphi \\ &=: I - II + III. \end{aligned}$$

The disjoint support part (III)

Since $\mu \leq \kappa < 2\mu$,

$$\begin{aligned}
\int P\Omega[P^T w_k] \varphi &\stackrel{(1.6)}{\lesssim} \|A\|_{2,B_r} \|\varphi\|_2 \|\mathcal{R}[P^T w_k]\|_{\infty,B_r} \\
&\stackrel{P.B.1}{\lesssim} \|A\|_{2,B_r} r^{\frac{m}{2}-\kappa+\mu} \|\nabla|^\tau \varphi\|_{\frac{m}{\kappa+\tau-\mu}} (2^k \Lambda r)^{-m+\kappa} \|w_k\|_{(p_\kappa, \infty)} \\
&\stackrel{(2.1)}{\lesssim} r^{\mu-\frac{m}{2}} \|A\|_{2,B_r} \|\nabla|^\tau \varphi\|_{\frac{m}{\kappa+\tau-\mu}} (2^k \Lambda)^{-m+\kappa} (2^k \Lambda r)^{m-2\mu} [w]_{(p_\kappa, \infty)_{\lambda_\kappa, A_{\Lambda, r}^k}} \\
&\stackrel{(2.15)}{\lesssim} \theta r^{m-2\mu} (2^k \Lambda)^{\kappa-2\mu} \|\nabla|^\tau \varphi\|_{\frac{m}{\kappa+\tau-\mu}} [w]_{(p_\kappa, \infty)_{\lambda_\kappa, A_{\Lambda, r}^k}}.
\end{aligned}$$

The same-support/commutator part (II)

We have

$$|II| \lesssim \|A\|_2 \|\mathcal{C}(\varphi, \mathcal{R})[P^T w_0]\|_{2,B_r} \stackrel{(2.15)}{\lesssim} r^{\frac{m-2\mu}{2}} \theta \|\mathcal{C}(\varphi, \mathcal{R})[P^T w_0]\|_{2,B_r}.$$

Now we apply Lemma 4.7, and have for arbitrary $\delta \in (0, 1)$, $\gamma_{1,2} \in (0, \delta)$,

$$|\mathcal{C}(\varphi, \mathcal{R})[P^T w_0]| \lesssim I_{\delta-\gamma_1} \|\nabla|^\delta \varphi\| I_{\gamma_1} |w_0| + C_{\mathcal{R}, \delta, \gamma_2} I_{\gamma_2} \left(\|\nabla|^\delta \varphi\| I_{\delta-\gamma_2} |w_0| \right)$$

Now, if we choose $\delta < \tau$

$$\|I_{\delta_1-\gamma_1} \|\nabla|^\delta \varphi\|_{(\frac{m}{\gamma_1+\kappa-\mu}, q)} \lesssim \|\nabla|^\tau \varphi\|_{(\frac{m}{\tau+\kappa-\mu}, q)},$$

and for $\beta < 2\mu - \kappa$, using [Ada75], see Lemma A.6,

$$r^{\frac{\lambda_\kappa-m}{p_{\gamma_1}}} \|I_\beta w_0\|_{(p_\beta, \infty), B_r} \lesssim \|I_\beta w_0\|_{(p_\beta, \infty)_{\lambda_\kappa}} \lesssim \|w_0\|_{(p_\kappa, \infty)_{\lambda_\kappa}}$$

where

$$\frac{1}{p_\beta} = \frac{m-\kappa}{m} \frac{2\mu-\kappa-\beta}{2\mu-\kappa} \in (0, 1).$$

Now,

$$\begin{aligned}
&\frac{1}{p_{\gamma_1}} + \frac{\gamma_1 + \kappa - \mu}{m} \\
&= \frac{\mu}{m} + (m-2\mu) \frac{(2\mu-\kappa) - \gamma_1}{m(2\mu-\kappa)} \\
&\leq \frac{1}{2},
\end{aligned} \tag{2.18}$$

if we choose $\gamma_1 \in (0, 2\mu - \kappa)$ as follows: If $\mu = \frac{m}{2}$ we can choose γ arbitrarily. If $\mu < \frac{m}{2}$ and $\mu \leq 1$, then we pick γ_1 sufficiently close to $2\mu - \kappa \leq 1$. That is, for any $\tau < \mu$ sufficiently close or greater than $2\mu - \kappa$ such that there is a $\gamma_1 < \delta < \tau$, $\delta < 2\mu - \kappa$, satisfying the above equation, we have

$$\|I_{\delta-\gamma_1} \|\nabla|^\delta \varphi\| I_{\gamma_1} |w_0|\|_{2,B_r} \lesssim r^{\frac{m}{2}-\frac{m}{p_{\gamma_1}}-(\gamma_1+\kappa-\mu)} r^{\frac{m-\lambda_\kappa}{p_{\gamma_1}}} \|\nabla|^\tau \varphi\|_{(\frac{m}{\tau+\kappa-\mu}, 2)} \|w_0\|_{(p_\kappa, \infty)_{\lambda_\kappa}},$$

and

$$\frac{m}{2} - \frac{m}{p_{\gamma_1}} - (\gamma_1 + \kappa - \mu) + \frac{m - \lambda_\kappa}{p_{\gamma_1}} = \frac{m}{2} - (\gamma_1 + \kappa - \mu) - (2\mu - \kappa - \gamma_1) = \frac{m}{2} - \mu.$$

As for the second term, for $\delta - \gamma_2 < 2\mu - \kappa$, using the formula (2.18) with δ instead of γ_1 ,

$$\begin{aligned}
\frac{1}{p_2} &:= \frac{\delta + \kappa - \mu}{m} + \frac{1}{p_{\delta-\gamma_2}} \\
&= \frac{\delta + \kappa - \mu}{m} + \frac{1}{p_\delta} + \gamma_2 \frac{m-\kappa}{m(2\mu-\kappa)} \leq \frac{1}{2} + \gamma_2 \frac{m-\kappa}{m(2\mu-\kappa)} < 1,
\end{aligned}$$

if we choose $\gamma_1 < \delta$ (as above γ_1) close enough $2\mu - \kappa$, and γ_2 very small. Consequently, if we set

$$\lambda := \lambda_\kappa,$$

and $\tilde{\lambda} \in (0, m)$ such that $\frac{\tilde{\lambda}-m}{p_2} = \frac{\lambda-m}{p\delta-\gamma_2}$, that is

$$\begin{aligned}
\frac{\tilde{\lambda}}{p_2} &= \frac{\lambda-m}{p\delta-\gamma_2} + \frac{m}{p_2} = \frac{\lambda_\kappa-m}{p\delta-\gamma_2} + \delta + \kappa - \mu + \frac{m}{p\delta-\gamma_2} \\
&= (\lambda_\kappa) \frac{m-\kappa}{m} \frac{2\mu-\kappa-(\delta-\gamma_2)}{2\mu-\kappa} + \delta + \kappa - \mu \\
&= \mu + \gamma_2
\end{aligned} \tag{2.19}$$

then

$$\begin{aligned}
\| |\nabla|^{\delta_2} \varphi I_{\delta_2-\gamma_2} |w_0| \|_{(p_2, 2)_{\tilde{\lambda}}} &\approx \sup_{B_\rho} \rho^{\frac{\tilde{\lambda}-m}{p_2}} \| |\nabla|^{\delta_2} \varphi I_{\delta_2-\gamma_2} |w_0| \|_{(p_2, 2)_{B_\rho}} \\
&\lesssim \| |\nabla|^\tau \varphi \|_{(\frac{m}{\tau+\kappa-\mu}, 2)} \sup_{B_\rho} \rho^{\frac{\tilde{\lambda}-m}{p_2}} \| I_{\delta_2-\gamma_2} |w_0| \|_{(p\delta_2-\gamma_2, \infty)_{B_\rho}} \\
&\approx \| |\nabla|^\tau \varphi \|_{(\frac{m}{\tau+\kappa-\mu}, 2)} \| I_{\delta_2-\gamma_2} |w_0| \|_{(p\delta_2-\gamma_2, \infty)_{\lambda, B_\rho}} \\
&\lesssim \| |\nabla|^\tau \varphi \|_{(\frac{m}{\tau+\kappa-\mu}, 2)} \| w_0 \|_{(p_\kappa, \infty)_{\lambda_\kappa}}.
\end{aligned}$$

Now observe

$$\begin{aligned}
&\frac{1}{2} - \left(\frac{1}{p_2} - \frac{\gamma_2}{\tilde{\lambda}} \right) \stackrel{(2.19)}{=} \frac{1}{2} - \frac{\mu}{p_2(\mu+\gamma_2)} \\
&= \frac{1}{2} - \frac{\mu}{\mu+\gamma_2} \left(\frac{\delta_2 + \kappa - \mu}{m} + \frac{m-\kappa}{m} \frac{2\mu-\kappa-(\delta_2-\gamma_2)}{2\mu-\kappa} \right) \\
&= \frac{1}{2} - \frac{\mu}{\mu+\gamma_2} \left(((2\mu-\kappa) - \delta_2) \frac{m-2\mu}{m(2\mu-\kappa)} + \frac{\mu}{m} + \frac{m-\kappa}{m} \frac{\gamma_2}{2\mu-\kappa} \right) \geq 0,
\end{aligned}$$

for sufficiently small γ_2 and δ_2 sufficiently close to $2\mu - \kappa$. In fact, this holds obviously, if $\frac{\mu}{m} < \frac{1}{2}$. If $\frac{\mu}{m} = \frac{1}{2}$, we have

$$\frac{\mu}{\mu+\gamma_2} \left(((2\mu-\kappa) - \delta_2) \frac{m-2\mu}{m(2\mu-\kappa)} + \frac{\mu}{m} + \frac{m-\kappa}{m} \frac{\gamma_2}{2\mu-\kappa} \right) = \frac{\mu}{\mu+\gamma_2} \left(\frac{1}{2} + \frac{\gamma_2}{2\mu} \right) = \frac{1}{2}$$

Moreover, one checks

$$\frac{m}{2} - \frac{\mu}{\mu+\gamma_2} \frac{m}{p_2} + \frac{\mu}{\mu+\gamma_2} \frac{m-\tilde{\lambda}}{p_2} = \frac{m}{2} - \frac{\mu}{\mu+\gamma_2} \frac{\tilde{\lambda}}{p_2} \stackrel{(2.19)}{=} \frac{m}{2} - \mu.$$

Thus,

$$\begin{aligned}
\| I_{\gamma_2} (|\nabla|^{\delta_2} \varphi I_{\delta_2-\gamma_2} |w_0|) \|_{2, B_r} &\lesssim r^{\frac{m}{2}-\mu} \| I_{\gamma_2} (|\nabla|^{\delta_2} \varphi I_{\delta_2-\gamma_2} |w_0|) \|_{(p_2, 2)_{\frac{\mu+\gamma_2}{\mu}, \tilde{\lambda}}} \\
&\lesssim r^{\frac{m}{2}-\mu} \| |\nabla|^{\delta_2} \varphi I_{\delta_2-\gamma_2} |w_0| \|_{(p_2, 2)_{\tilde{\lambda}}} \\
&\lesssim r^{\frac{m}{2}-\mu} \| |\nabla|^\tau \varphi \|_{(\frac{m}{\tau+\kappa-\mu}, 2)} \| w_0 \|_{(p_\kappa, \infty)_{\lambda_\kappa}}.
\end{aligned}$$

The same-support/commutator part (I)

Here, we decompose

$$w_0 \varphi = |\nabla|^\mu (\eta_{\Lambda r} (I_\mu(w_0 \varphi))) + |\nabla|^\mu ((1 - \eta_{\Lambda r}) (I_\mu(w_0 \varphi))) =: |\nabla|^\mu g_1 + |\nabla|^\mu g_2$$

and

$$\begin{aligned}
I &= \int (|\nabla|^\mu P) P^T |\nabla|^\mu g_1 + P \Omega [P^T |\nabla|^\mu g_1] + \int (|\nabla|^\mu P) P^T |\nabla|^\mu g_2 + P \Omega [P^T |\nabla|^\mu g_2] \\
&=: I_1 + I_2.
\end{aligned}$$

For I_1 we use Theorem 1.6 in the form of Lemma 5.7,

$$I_1 = \int \Omega_P[|\nabla|^\mu g_1] \lesssim \theta r^{m-2\mu} \begin{cases} [g_1]_{\text{BMO}} & \text{if } \mu \leq 1, \\ r^{\mu-\frac{m}{2}} \| |\nabla|^\mu g_1 \|_{(2,\infty)} & \text{if } \mu > 1. \end{cases}$$

Note that

$$\text{supp}(\varphi w_0) \subset B_r,$$

and moreover for $q_\mu = \infty$, for $\kappa > \mu$, and $q_\mu = 1$ for $\kappa = \mu$, (for arbitrary $\tau > 0$)

$$\|\varphi w_0\|_{(\frac{m}{m-\mu}, \infty)_{\frac{\mu m}{m-\mu}}} \lesssim \|\varphi\|_{(\frac{m}{\kappa-\mu}, \infty)} \|w\chi_{B_r}\|_{(p_\kappa, \infty)_{\lambda_\kappa}} \lesssim \| |\nabla|^\tau \varphi \|_{(\frac{m}{\kappa+\tau-\mu}, q_\mu)} \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_{2r}}. \quad (2.20)$$

Then, the claim for I_1 follows from

Proposition 2.4. *Let $\mu \leq 1$, $g := \eta_{\Lambda r} I_\mu(f)$, $\text{supp } f \subset \overline{B_r}$, then for any $\kappa \in [\mu, 2\mu)$,*

$$[g]_{\text{BMO}} \lesssim (1 + \Lambda^{\mu-m}) \|f\|_{(\frac{m}{m-\mu}, \infty)_{\frac{\mu m}{m-\mu}}}.$$

Proof. From [Ada75, Proposition 3.3.]

$$[g]_{\text{BMO}} \lesssim \| |\nabla|^\mu g \|_{(1)_\mu}.$$

Since,

$$|\nabla|^\mu g = f + |\nabla|^\mu((1 - \eta_{\Lambda r})I_\mu f),$$

we have,

$$[g]_{\text{BMO}} \lesssim \|f\|_{(1)_\mu} + \| |\nabla|^\mu((1 - \eta_{\Lambda r})I_\mu f) \|_{(1)_\mu} \lesssim \|f\|_{(\frac{m}{m-\mu}, \infty)_{\frac{\mu m}{m-\mu}}} + \| |\nabla|^\mu((1 - \eta_{\Lambda r})I_\mu f) \|_{(\frac{m}{\mu}, \infty)},$$

and by Proposition B.3

$$\| |\nabla|^\mu((1 - \eta_{\Lambda r})I_\mu f) \|_{\frac{m}{\mu}} \lesssim \sup_{\alpha \in [0, \mu]} (\Lambda r)^{-m+\mu-\alpha} \|f\|_1 \| |\nabla|^{\mu-\alpha}((1 - \eta_{\Lambda r})) \|_{\frac{m}{\mu}} \lesssim (\Lambda r)^{\mu-m} \|f\|_1. \quad (2.21)$$

Since $\text{supp } f \subset B_r$,

$$r^{\mu-m} \|f\|_1 \lesssim \|f\|_{(1)_\mu} \lesssim \|f\|_{(\frac{m}{m-\mu}, \infty)_{\frac{\mu m}{m-\mu}}}. \quad (2.22)$$

□

Moreover, as in (2.21), from Proposition B.3 and (2.20),

$$\| |\nabla|^\mu g_2 \|_2 \lesssim (\Lambda r)^{-\frac{m}{2}} \|\varphi w_0\|_1 \stackrel{(2.22)}{\lesssim} r^{\frac{m}{2}-\mu} \| |\nabla|^\tau \varphi \|_{(\frac{m}{\kappa+\tau-\mu}, q_\mu)} \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_{2r}},$$

implying

$$\begin{aligned} |I_2| &\lesssim \|\Omega_P\|_{2 \rightarrow 1} (\Lambda r)^{\frac{m}{2}-\mu} \| |\nabla|^\tau \varphi \|_{(\frac{m}{\kappa+\tau-\mu}, q_\mu)} \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_{2r}} \\ &\stackrel{(2.17)}{\lesssim} \theta r^{m-2\mu} \| |\nabla|^\tau \varphi \|_{(\frac{m}{\kappa+\tau-\mu}, q_\mu)} \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_{2r}}. \end{aligned}$$

This proves the claim (2.13) and thus Lemma 2.3

3. Higher Integrability: Proof of Theorem 1.3

Let $w \in L_{loc}^{p,\lambda}(D) \cap L^2(\mathbb{R}^m)$ be a solution to

$$|\nabla|^\mu w = \Omega[w] \quad \text{in } D \subset\subset \mathbb{R}^m.$$

Choosing for any domain $\tilde{D} \subset\subset D$, we can choose a domain D_2 , $\tilde{D} \subset\subset D_2 \subset\subset D$ and a cutoff function $\eta_{\tilde{D}} \in C_0^\infty(D_2)$, $\eta_{\tilde{D}} \equiv 1$ in \tilde{D} . Then $w_{\tilde{D}} := \eta_{\tilde{D}} w \in L^{p,\lambda}(\mathbb{R}^n)$ is a solution to

$$|\nabla|^\mu w_{\tilde{D}} = \Omega[w_{\tilde{D}}] + \Omega[w - w_{\tilde{D}}] + |\nabla|^\mu(w_{\tilde{D}} - w) \quad \text{in } \tilde{D},$$

and in \tilde{D} ,

$$\|\Omega[w - w_{\tilde{D}}] + |\nabla|^\mu(w_{\tilde{D}} - w)\|_{\infty, \tilde{D}} \leq C_{\tilde{D}, D, D_2, \eta, \|w\|_2}.$$

So Theorem 1.3 follows from the following argument.

Lemma 3.1. *Let $p > 2$, and $0 < \mu \leq \frac{m}{2}$, $\lambda \leq 2\mu$, and let $w \in L^{p,\lambda}$ be a solution to*

$$|\nabla|^\mu w = \Omega[w] + f \quad \text{in } D \subset\subset \mathbb{R}^m, \quad (3.1)$$

where $f \in L^\infty(D)$. Then, for any $\tilde{p} \in [p, \infty)$ there exists $\varepsilon \in (0, 1)$ such that if θ from (1.8) satisfies $\theta < \varepsilon$, then $w \in L_{loc}^{\tilde{p}}(D)$.

Proof. In order to keep the presentation short, we are going to assume that $\Omega[\cdot] = \tilde{\Omega}\mathcal{R}[\cdot]$. Also note that if $w \in L^{p,\lambda}$ for some $p > 2$, then for some $\tilde{p} \in (2, p)$, $w \in L^{\tilde{p}, \tilde{\lambda}}$, for some $\tilde{\lambda} < \lambda$, so we can assume w.l.o.g. that $\lambda < 2\mu$. From (3.1) we have for any $B_r \subset B_R \subset \tilde{D}$,

$$\begin{aligned} \|\nabla|^\mu w\|_{\frac{2p}{p+2}, B_r} &\lesssim \|\Omega\|_{2, B_r} \|\mathcal{R}[w]\|_{p, B_r} + \|f\|_\infty r^{m\frac{p+2}{2p}} \\ &\stackrel{(1.8)}{\lesssim} r^{\frac{m-2\mu}{2}} \theta \|w\|_{p, B_{2r}} + r^{\frac{m-2\mu}{2}} \theta \sum_{k=2}^{\infty} 2^{-k\frac{m}{p}} \|w\|_{p, A_r^k} + \|f\|_\infty r^{m\frac{p+2}{2p}} \\ &\lesssim r^{\frac{m-2\mu}{2}} r^{\frac{m-\lambda}{p}} \theta \|w\|_{(p)\lambda, B_R} + r^{\frac{m-2\mu}{2}} r^{\frac{m-\lambda}{p}} \theta \sum_{k=2}^{\infty} 2^{-k\frac{\lambda}{p}} [w]_{(p)\lambda, B_{2^{k+1}R}} + \|f\|_\infty r^{m\frac{p+2}{2p}} \end{aligned}$$

That is, for

$$\lambda_N := \lambda \frac{2}{2+p} + 2\mu \frac{p}{2+p} \in (\lambda, 2\mu), \quad (3.2)$$

$$\|\nabla|^\mu w\|_{(\frac{2p}{p+2})\lambda_N, B_{\frac{R}{2}}} \lesssim \theta \|w\|_{(p)\lambda, B_R} + \theta \sum_{k=2}^{\infty} 2^{-k\frac{\lambda}{p}} [w]_{(p)\lambda, B_{2^{k+1}R}} + \|f\|_\infty R^{\lambda_N \frac{p+2}{2p}}$$

Consequently, by Proposition 3.2 (note that $\frac{2p}{p+2} > 1$), for $p_2 = 2p/(p+2)$ and $p_1 > p$ (since $\lambda_N < 2\mu$) defined by

$$\frac{1}{p_1} = \frac{1}{p} + \frac{1}{2} - \frac{\mu}{\lambda_N} \quad (3.3)$$

$$\begin{aligned} \|w\|_{p_1, B_{\Lambda^{-1}r}} &\lesssim (\Lambda^{-1}r)^{-\frac{\lambda_N}{p} - \frac{\lambda_N}{2} + \mu + \frac{m}{p_1}} \|\nabla|^\mu w\|_{(\frac{2p}{p+2})\lambda_N, B_r} + \Lambda^{-\frac{m}{p_1}} \sum_{k=1}^{\infty} 2^{-km} r^{\frac{m}{p_1} - m} \|w\|_{1, A_r^k} \\ &\lesssim (\Lambda^{-1}r)^{-\frac{\lambda_N}{p} - \frac{\lambda_N}{2} + \mu + \frac{m}{p_1}} \theta \|w\|_{(p)\lambda, B_{2r}} + (\Lambda^{-1}r)^{-\frac{\lambda_N}{p} - \frac{\lambda_N}{2} + \mu + \frac{m}{p_1}} \theta \sum_{k=2}^{\infty} 2^{-k\frac{\lambda}{p}} [w]_{(p)\lambda, B_{2^{k+2}r}} \\ &\quad + (\Lambda^{-1}r)^{-\frac{\lambda_N}{p} - \frac{\lambda_N}{2} + \mu + \frac{m}{p_1}} \|f\|_\infty r^{\lambda_N \frac{p+2}{2p}} \\ &\quad + (\Lambda^{-1}r)^{\frac{m}{p_1} - \frac{\lambda}{p}} \Lambda^{-\frac{\lambda}{p}} \sum_{k=1}^{\infty} 2^{-k\frac{\lambda}{p}} [w]_{(p)\lambda, B_{2^{k+1}r}} \\ &\lesssim (\Lambda^{-1}r)^{-\frac{\lambda_N}{p} - \frac{\lambda_N}{2} + \mu + \frac{m}{p_1}} \theta \|w\|_{(p)\lambda, B_{2r}} \\ &\quad + (\Lambda^{-1}r)^{-\frac{\lambda_N}{p} - \frac{\lambda_N}{2} + \mu + \frac{m}{p_1}} (\theta + \Lambda^{-\frac{\lambda}{p}}) \sum_{k=1}^{\infty} 2^{-k\frac{\lambda}{p}} [w]_{(p)\lambda, B_{2^{k+1}r}} \\ &\quad + (\Lambda^{-1}r)^{-\frac{\lambda_N}{p} - \frac{\lambda_N}{2} + \mu + \frac{m}{p_1}} \|f\|_\infty r^{\lambda_N \frac{p+2}{2p}} \end{aligned}$$

Consequently,

$$\begin{aligned}
\|w\|_{p, B_{\Lambda^{-1}r}} &\lesssim (\Lambda^{-1}r)^{\frac{m}{p} - \frac{m}{p_1}} \|w\|_{p_1, B_{\Lambda^{-1}r}} \\
&\lesssim (\Lambda^{-1}r)^{\frac{m}{p} - \frac{\lambda_N}{p} - \frac{\lambda_N}{2} + \mu} \theta \|w\|_{(p)\lambda, B_{2r}} \\
&\quad + (\Lambda^{-1}r)^{\frac{m}{p} - \frac{\lambda_N}{p} - \frac{\lambda_N}{2} + \mu} (\theta + \Lambda^{-\frac{\lambda}{p}}) \sum_{k=1}^{\infty} 2^{-k\frac{\lambda}{p}} [w]_{(p)\lambda, B_{2^{k+1}r}} \\
&\quad + (\Lambda^{-1}r)^{\frac{m}{p} - \frac{\lambda_N}{p} - \frac{\lambda_N}{2} + \mu} \|f\|_{\infty} r^{\lambda_N \frac{p+2}{2p}}.
\end{aligned}$$

Now

$$\frac{m}{p} - \frac{\lambda_N}{p} - \frac{\lambda_N}{2} + \mu = \frac{m - \lambda}{p},$$

which implies finally, for any $B_{2r} \subset D$,

$$\begin{aligned}
\|w\|_{(p)\lambda, B_{\Lambda^{-1}r}} &\lesssim \theta \|w\|_{(p)\lambda, B_{2r}} \\
&\quad + (\theta + \Lambda^{-\frac{\lambda}{p}}) \sum_{k=1}^{\infty} 2^{-k\frac{\lambda}{p}} [w]_{(p)\lambda, B_{2^{k+1}r}} \\
&\quad + \|f\|_{\infty} r^{\lambda_N \frac{p+2}{2p}}.
\end{aligned}$$

Now we argue similar to the iteration in Section 2: Choose $\Lambda_{\lambda} := 2^{C_{p,\mu}\lambda^{-4}}$, assume that $\theta < \Lambda_{\lambda}^{-\frac{\lambda}{p}}$, and choose $C_{p,\mu}$ so that (D.1) is satisfied. Then we can choose a new $\lambda_1 = \lambda - c\lambda^4$ for which the above estimate holds and the right-hand side is finite. Repeating this argument (for smaller and smaller θ), we obtain a monotone decreasing sequence of $\lambda_{i+1} = \lambda_i - c\lambda_i^4 \geq 0$, which has as only fixed point 0. Thus, for any $\lambda > 0$ there exists $\theta > 0$ such that for any $\tilde{D} \subset\subset D$,

$$\|w\|_{(p)\lambda, \tilde{D}} \leq C_{\tilde{D}, D, \lambda, w}.$$

Note that for $\lambda \rightarrow 0$, $\lambda_N \rightarrow \mu \frac{2p}{p+2}$ and thus p_1 in (3.3) tends to infinity. Thus, we have obtain for any $\tilde{p} > 1$ a $\lambda_{\tilde{p}} > 0$ such that $p_1 \equiv p_1(\lambda_{\tilde{p}}) > \tilde{p}$, and if θ is small enough, we have to iterate the above argument finitely many steps to obtain that $w \in L_{loc}^{p_1}(\tilde{D})$. \square

Proposition 3.2. *For any f , $\mu \in (0, m)$ we have for $p_1 \in (1, \infty)$, $p_2 \in (1, \infty)$, $\lambda \in (0, m)$ such that*

$$\frac{1}{p_1} = \frac{1}{p_2} - \frac{\mu}{\lambda},$$

the following estimate for any $\Lambda > 2$

$$\|f\|_{p_1, B_{\Lambda^{-1}r}} \lesssim (\Lambda^{-1}r)^{-\frac{\lambda}{p_2} + \mu + \frac{m}{p_1}} \|\nabla|^{\mu} f\|_{(p_2)\lambda, B_r} + \sum_{k=1}^{\infty} 2^{-km} \Lambda^{-\frac{m}{p_1}} r^{\frac{m}{p_1} - m} \|f\|_{1, A_r^k}$$

Proof. Let $1 < p_4 \leq p'_1$,

$$\begin{aligned}
\frac{1}{p_3} + \frac{1}{p_4} &= 1. \\
\frac{1}{p_3} &= \frac{1}{p_2} - \frac{\mu}{\lambda} \in (0, 1).
\end{aligned}$$

There exists $\varphi \in C_0^{\infty}(B_{\Lambda^{-1}r})$, $\|\varphi\|_{p'_1} \leq 1$, such that

$$\begin{aligned}
\|f\|_{p_1, B_{\Lambda^{-1}r}} &\lesssim \int f \varphi = \int I_{\mu}(\eta_{B_r} |\nabla|^{\mu} f) \varphi + \sum_{k=1}^{\infty} \int f |\nabla|^{\mu} (\eta_{A_r^k} I_{\mu} \varphi) \\
&\lesssim \|I_{\mu}(\eta_r |\nabla|^{\mu} f)\|_{p_3, B_{\Lambda^{-1}r}} \|\varphi\|_{p_4} + \sum_{k=1}^{\infty} \|f\|_{p_1, A_r^k} \|\nabla|^{\mu} (\eta_{A_r^k} I_{\mu} \varphi)\|_{p'_1}
\end{aligned}$$

By Lemma B.3,

$$\|\nabla|^{\mu} (\eta_{A_r^k} I_{\mu} \varphi)\|_{\infty} \lesssim 2^{-km} \Lambda^{-\frac{m}{p_1}} r^{\frac{m}{p_1} - m}$$

Since $p_4 \leq p'_1$,

$$\|\varphi\|_{p_4, B_r} \lesssim (\Lambda^{-1}r)^{\frac{m}{p_4} - \frac{m}{p'_1}}.$$

And using Lemma A.6

$$\|I_\mu(\eta_r|\nabla|^\mu f)\|_{p_3, B_{\Lambda^{-1}r}} \lesssim (\Lambda^{-1}r)^{\frac{m-\lambda}{p_3}} \|I_\mu(\eta_r|\nabla|^\mu f)\|_{(p_3)_\lambda} \lesssim (\Lambda^{-1}r)^{\frac{m-\lambda}{p_3}} \|\nabla|^\mu f\|_{(p_2)_\lambda, B_r}$$

Consequently, we have shown the claim. □

4. Commutators and fractional product rules: Proof of Theorem 1.4

In this section we repeat and refine some commutator estimates and non-local expansion rules which were introduced in [Sch11], motivated by the results in [DLR11b, Sch12]. The for us most important commutators are

$$H_\alpha(a, b) := |\nabla|^\alpha(ab) - a|\nabla|^\alpha b - b|\nabla|^\alpha a,$$

and for a linear operator T

$$\mathcal{C}(a, T)[b] := aT[b] - T[ab].$$

The commutator $H_\alpha(a, b)$ was introduced by Da Lio and Rivière in [DLR11b], where Hardy-space \mathcal{H} and BMO -estimates were shown, making use of the Hardy-Littlewood decomposition and paraproducts. This is also somewhat related to the techniques of the T1-Theorem cf. [KP88]. If one is interested in L^2 -estimates only (e.g., in the sphere case) then there is an extremely elementary argument [Sch12] somewhat inspired by Tartar's proof of Wente's inequality [Tar85]. For general Lorentz space estimates there is also an argument using potential arguments, which even gives pointwise estimates, and was introduced in [Sch11]. As it is a direct, pointwise argument not involving the Fourier transform, it is easier to apply in non-linear situations, cf. [BRS12].

The commutator $\mathcal{C}(a, T)[b]$ and its Hardy-BMO estimates were introduced in [CRW76] for the Riesz transform \mathcal{R} , and later generalized to the Riesz potential I_α in [Cha82]. Again for pointwise estimates the arguments in [Sch12] can be adapted.

Here, we are going to treat in Subsection 4.1 pointwise estimates on $H_\alpha(a, b)$, and in Subsection 4.2 pointwise estimates on $\mathcal{C}(a, T)[b]$ using and extending the techniques from [Sch12]. For Hardy-BMO estimates, we will use in Subsection 4.3 the techniques in [DLR11b], and extend them to the limiting case $\alpha = 1$.

Let us shortly recall the notion for Hardy space \mathcal{H} and BMO . The latter space BMO is defined as

$$g \in BMO \quad \Leftrightarrow [g]_{BMO} := |B_r|^{-1} \int_{B_r} \left| g - |B_r|^{-1} \int_{B_r} g \right| < \infty.$$

Our interest in BMO stems from the fact, that it is a bigger space than L^∞ , and we have the nice embedding

$$[g]_{BMO} \lesssim \sup_{r>0} r^{\frac{p\tau-m}{p}} \| |\nabla|^\tau g \|_{(p,\infty), B_r} \quad \text{for } \tau > 0, p > 1, \quad (4.1)$$

whereas for L^∞ we only have the following embedding which is more difficult to control

$$\|g\|_\infty \lesssim \| |\nabla|^\tau g \|_{(\frac{m}{\tau}, 1)} \quad \text{for } \tau \in (0, m). \quad (4.2)$$

The Hardy space \mathcal{H} , on the other hand, is a slightly smaller space than L^1 , with the (for us) most important property that

$$\int f g \lesssim \|f\|_{\mathcal{H}} [g]_{BMO}. \quad (4.3)$$

That is, if we know that a quantity belongs to the Hardy space, it allows us to control the integral of (4.3) in terms of the right-hand side of (4.1), instead of having to deal with the terms on the right-hand side of (4.2).

The norm of the Hardy space \mathcal{H} is usually defined via

$$\|f\|_{\mathcal{H}} := \left\| \sup_{t>0} \phi_t * f \right\|_1,$$

where $\phi \in C_0^\infty(B_1)$, $\int \phi = 1$, and $\phi_t(x) := t^{-m} \phi(x/t)$, cf. [Ste93, FS72], another very readable overview in the context with Partial Differential Equations is given in [Sem94]. We are never using the above definition, though, but rather use a characterization in terms of Triebel-spaces, and employ the duality (4.3).

4.1. Pointwise fractional product rules via potentials

Lemma 4.1. *For any $\alpha \in (0, m)$ there is $L \in \mathbb{N}$ such that the following holds: For any $\beta \in [0, \min(\alpha, 1))$, $\beta \leq m - \alpha$, $\tau \in (\max\{\beta, \alpha + \beta - 1\}, \alpha]$, $\epsilon > 0$, there are, $s_k \in (0, \alpha)$, $t_k \in (0, \tau)$, where $\tau - \beta - s_k - t_k \in [0, \epsilon)$, such that the following holds*

$$\left| |\nabla|^\beta H_\alpha(a, b) \right| \lesssim \sum_{k=1}^L I_{\tau-\beta-s_k-t_k} (I_{s_k} | |\nabla|^\alpha a | I_{t_k} | |\nabla|^\tau b |).$$

Proof. Notice that it suffices to prove the for $\tau < \alpha$, given that

$$||\nabla|^\tau b| = |I_{\alpha-\tau}|\nabla|^\alpha b| \lesssim I_{\alpha-\tau}||\nabla|^\alpha b|. \quad (4.4)$$

$$\begin{aligned} |\nabla|^\beta H_\alpha(a, b) &= |\nabla|^\beta (|\nabla|^\alpha(ab) - a|\nabla|^\alpha b - b|\nabla|^\alpha a) \\ &= |\nabla|^{\alpha+\beta}(ab) - a|\nabla|^{\alpha+\beta}b - b|\nabla|^{\alpha+\beta}a \\ &\quad + a|\nabla|^\beta|\nabla|^\alpha b - |\nabla|^\beta(a|\nabla|^\alpha b) \\ &\quad + b|\nabla|^\beta|\nabla|^\alpha a - |\nabla|^\beta(b|\nabla|^\alpha a) \\ &= H_{\alpha+\beta}(a, b) + \mathcal{C}(a, |\nabla|^\beta)[|\nabla|^\alpha b] + \mathcal{C}(b, |\nabla|^\beta)[|\nabla|^\alpha a] \end{aligned}$$

The claim for the term $H_{\alpha+\beta}(a, b)$ then comes from Lemma 4.2.

For the remaining terms, we apply Lemma 4.5: For $\mathcal{C}(a, |\nabla|^\beta)[|\nabla|^\alpha b]$, we take $\delta \hat{=} \alpha - \tau$, $\tau \hat{=} \alpha$, $\beta \hat{=} \beta$ and set $A := |\nabla|^\alpha a$, $B := |\nabla|^\tau b$. Then

$$\mathcal{C}(a, |\nabla|^\beta)[|\nabla|^\alpha b] = \mathcal{C}(I_\alpha A, |\nabla|^\beta)[|\nabla|^{\alpha-\tau} B].$$

For $\mathcal{C}(b, |\nabla|^\beta)[|\nabla|^\alpha a]$, set for very small $\delta < \min\{\tau - \beta, 1 - \beta\}$, $A := |\nabla|^{\alpha-\delta} a$, $B := |\nabla|^\tau b$. Then

$$\mathcal{C}(b, |\nabla|^\beta)[|\nabla|^\alpha a] = \mathcal{C}(I_\tau B, |\nabla|^\beta)[|\nabla|^\delta A].$$

□

Lemma 4.2. *Let $\alpha \in (0, m)$, $\epsilon > 0$ and assume that $\tau_1, \tau_2 \in (\max\{\alpha - 1, 0\}, \alpha]$, $\tau_1 + \tau_2 > \alpha$. Then for some $L \in \mathbb{N}$, there are $s_k \in (0, \tau_1)$, $t_k \in (0, \tau_2)$, $\tau_1 + \tau_2 - s_k - t_k - \alpha \in [0, \epsilon)$ such that*

$$|H_\alpha(a, b)| \lesssim \sum_{k=1}^L I_{\tau_1+\tau_2-s_k-t_k-\alpha}(I_{s_k}||\nabla|^{\tau_1} a| I_{t_k}||\nabla|^{\tau_2} b|). \quad (4.5)$$

For the convenience of the reader, we give the proof, which essentially follows the argument in [Sch11]; For a presentation closer to this one see [DLS12a].

Proof. For $\alpha \in 2\mathbb{N}$ it is easy to obtain (4.5), since for any $l \leq 2K - 1$, for $\tau > 2K - 1$,

$$|\nabla^l f| = |\nabla^l I_\tau |\nabla|^\tau f| \lesssim I_{\tau_2-l} ||\nabla|^\tau f|. \quad (4.6)$$

So we can assume that $\alpha = 2K + s$, for some $s \in (0, 2)$, $K \in \mathbb{N} \cup \{0\}$. Assume at first that $K = 0$. Set $A := ||\nabla|^{\tau_1} a|$, $B := ||\nabla|^{\tau_2} b|$, we have

$$|H_\alpha(a, b)| \lesssim \lim_{\epsilon \rightarrow 0} \int_{|x-y|>\epsilon} \int \int k(x, y, z_1, z_2) A(z_1) B(z_2) dz_2 dz_1 dy,$$

where

$$k(x, y, z_1, z_2) = \frac{m_{\tau_1}(x, y, z_1) m_{\tau_2}(x, y, z_2)}{|x-y|^{m+\alpha}},$$

and for $s > 0$,

$$m_s(x, y, z) = \left| |x-z|^{-m+s} - |z-y|^{-m+s} \right|.$$

Let moreover

$$1 \leq \chi_1(x, y, z) + \chi_2(x, y, z) + \chi_3(x, y, z) \quad \text{for } x, y, z \in \mathbb{R}^m,$$

where

$$\chi_1 := \chi_{|x-y| \leq 2|z-y|} \chi_{|x-y| \leq 2|x-z|},$$

$$\chi_2 := \chi_{|x-y| \leq 2|z-y|} \chi_{|x-y| > 2|x-z|},$$

$$\chi_3 := \chi_{|x-y| > 2|z-y|} \chi_{|x-y| \leq 2|x-z|}.$$

One then checks, using for $m_s(x, y, z)\chi_1$ a one-step Taylor expansion, for any $\delta \in (0, \min(s, 1))$

$$\begin{aligned} m_s(x, y, z)\chi_1 &\lesssim |x - z|^{-n+\alpha-\delta}|x - y|^\delta \chi_1 \approx |z - y|^{-n+\alpha-\delta}|x - y|^\delta \chi_1. \\ m_s(x, y, z)\chi_2 &\lesssim |x - z|^{-n+\alpha}\chi_2 \lesssim |x - z|^{-n+\alpha-\delta}|x - y|^\delta, \\ m_s(x, y, z)\chi_3 &\lesssim |z - y|^{-n+\alpha}\chi_3 \lesssim |z - y|^{-n+\alpha-\delta}|x - y|^\delta. \end{aligned}$$

Hence, for $\delta_1 \in (0, \min(\tau_1, 1))$, $\delta_2 \in (0, \min(\tau_2, 1))$

$$\begin{aligned} k(x, y, z_1, z_2) &\lesssim |x - y|^{-m-\alpha+\delta_1+\delta_2} \left(|x - z_1|^{-m+\tau_1-\delta_1} |y - z_2|^{-m+\tau_2-\delta_2} \right. \\ &\quad + |y - z_1|^{-m+\tau_1-\delta_1} |x - z_2|^{-m+\tau_2-\delta_2} \\ &\quad + |y - z_1|^{-m+\tau_1-\delta_1} |y - z_2|^{-m+\tau_2-\delta_2} \\ &\quad \left. + |x - y|^{-\delta_1-\delta_2}|x - z_1|^{-m+\tau_1} |x - z_2|^{-m+\tau_2}\chi_{|x-y|>2\max\{|x-z_1|, |x-z_2|\}} \right). \end{aligned}$$

We choose $\delta_1 \in (0, \min(\tau_1, 1))$, $\delta_2 \in (0, \min(\tau_2, 1))$ such that $\delta_1 + \delta_2 - \alpha \in (0, \epsilon)$. This is possible, since $\tau_1 + \tau_2 > \alpha$: If $\alpha \in (0, 1]$, so are τ_1, τ_2 , and we can choose δ_1, δ_2 arbitrarily close to τ_1, τ_2 , so that this inequality is satisfied. If $\alpha \in [1, 2)$ and (say) $\tau_1 > 1$, choose δ_1 close enough to 1, and $\delta_2 \in (\alpha - 1, \tau_2)$. Using that

$$\int_{|x-y|>\epsilon} |x - y|^{-m-\alpha}\chi_{|x-y|>2\max\{|x-z_1|, |x-z_2|\}} dy \lesssim \max\{|x - z_1|, |x - z_2|\}^{-\alpha} \lesssim |x - z_1|^{-\delta_1} |x - z_2|^{-\alpha+\delta_1},$$

and consequently,

$$\begin{aligned} H_\alpha(a, b) &\lesssim I_{\alpha-\delta_1}A (I_{\delta_1+\delta_2-\alpha}I_{\tau_2-\delta_2}B) \\ &\quad + (I_{\delta_1+\delta_2-\alpha}I_{\alpha-\delta_1}A) I_{\tau_2-\delta_2}B \\ &\quad + I_{\delta_1+\delta_2-\alpha}(I_{\alpha-\delta_1}A I_{\tau_2-\delta_2}B) \\ &\quad + I_{\alpha-\delta_1}A I_{\tau_2+\delta_1-\alpha}B \\ &\approx I_{\alpha-\delta_1}A I_{\delta_1-\alpha+\tau_2}B \\ &\quad + I_{\delta_1+\delta_2-\alpha}(I_{\alpha-\delta_1}A I_{\tau_2-\delta_2}B) \\ &\quad + I_{\delta_2}A I_{\tau_2-\delta_2}B. \end{aligned}$$

This shows (4.5) for $\alpha \in (0, 2)$.

If $K \geq 1$, $s \in (0, 2)$, $\alpha = 2K + s > 2$

$$\begin{aligned} H_{2K+s}(a, b) &= \Delta^K |\nabla|^s(ab) - a \Delta^K |\nabla|^s b - b \Delta^K |\nabla|^s a \\ &= |\nabla|^s(\Delta^K(ab) - a\Delta^K b - b\Delta^K a) \end{aligned} \tag{4.7}$$

$$+ |\nabla|^s(b\Delta^K a) - b \Delta^K |\nabla|^s a \tag{4.8}$$

$$+ |\nabla|^s(a\Delta^K b) - a \Delta^K |\nabla|^s b. \tag{4.9}$$

Let ∇^l, ∇^{2K-l} , $l \in 1, \dots, 2K - 1$, be arbitrary combinations of gradients which sum up to differential order of l and $2K - l$, respectively. Then

$$|\nabla|^s(\nabla^l a \nabla^{2K-l} b) = H_s(\nabla^l a, \nabla^{2K-l} b) + |\nabla|^s \nabla^l a \nabla^{2K-l} b + \nabla^l a |\nabla|^s \nabla^{2K-l} b.$$

Recall $\alpha = 2K + s$. Applying (4.5) to $H_s(\cdot, \cdot)$, noting that $\alpha - s - l = 2K - l > 0$ and $\tau_2 - s - 2K + l > l - 1 \geq 0$

$$\begin{aligned}
|\nabla|^s (\nabla^k a \nabla^{2K-k} b) &\lesssim \sum_{k=1}^L I_{s-s_k-t_k} (I_{s_k} |\nabla|^s \nabla^l a| I_{t_k} |\nabla|^s \nabla^{2K-l} b|) \\
&\quad + |\nabla|^s \nabla^l a| |\nabla^{2K-l} b| + |\nabla^l a| |\nabla|^s \nabla^{2K-l} b| \\
&\stackrel{(4.6)}{\lesssim} \sum_{k=1}^L I_{s-s_k-t_k} (I_{s_k+\tau_1-s-l} |\nabla|^{\tau_1} a| I_{t_k-\alpha+l+\tau_2} |\nabla|^{\tau_2} b|) \\
&\quad + I_{\tau_1-s-l} |\nabla|^{\tau_1} a| I_{\tau_2-2K+l} |\nabla|^{\tau_2} b| + I_{\tau_1-l} |\nabla|^{\tau_1} a| I_{\tau_2-\alpha+l} |\nabla|^{\tau_2} b|.
\end{aligned}$$

Thus, we can estimate (4.7)

$$\begin{aligned}
|\nabla|^s (\Delta^K(ab) - a\Delta^K b - b\Delta^K a) &\lesssim \sum_{l=1}^{2K-1} \sum_{k=1}^L I_{s-s_k-t_k} (I_{s_k+\tau_1-s-l} |\nabla|^{\tau_1} a| I_{t_k-\alpha+l+\tau_2} |\nabla|^{\tau_2} b|) \\
&\quad + \sum_{l=1}^{2K-1} I_{\tau_1-s-l} |\nabla|^{\tau_1} a| I_{\tau_2-2K+l} |\nabla|^{\tau_2} b| + \sum_{l=1}^{2K-1} I_{\tau_1-l} |\nabla|^{\tau_1} a| I_{\tau_2-\alpha+l} |\nabla|^{\tau_2} b|.
\end{aligned}$$

As for (4.8),

$$|\nabla|^s (b\Delta^K a) - b \Delta^K |\nabla|^s a = H_s(\Delta^K a, b) + \Delta^K a |\nabla|^s b.$$

We have $\tau_1, \tau_2 > \alpha - 1 = 2K + s - 1$, and consequently ($K \geq 1$) we know $\tau_2 > s$. Assume that moreover $\tau_1 > 2K$, then

$$|\nabla|^{2K} a| \lesssim I_{\tau_2-2K} |\nabla|^{\tau_1} a|, \quad |\nabla|^s b| \lesssim I_{\tau_2-s} |\nabla|^{\tau_2} b|,$$

and applying our estimates on $H_s(\cdot, \cdot)$ for $\tilde{\tau}_1 = \tau_1 - 2K > s - 1$ (since $\tau_1 > \alpha - 1$) and $\tilde{\tau}_2 = s$ we have the claim. If this is not the case and $\tau_1 \leq 2K$, then $s < 1$. Then we need to apply Lemma 4.5:

$$\begin{aligned}
|\nabla|^s (b\Delta^K a) - b \Delta^K |\nabla|^s a &= |\nabla|^s (b |\nabla|^\delta I_{\tau_1-2K+\delta} |\nabla|^{\tau_1} a) - b |\nabla|^s |\nabla|^\delta I_{\tau_1-2K+\delta} |\nabla|^{\tau_1} a \\
&= \mathcal{C}(I_{\tau_2} B, |\nabla|^s) [|\nabla|^\delta A],
\end{aligned}$$

for $B := |\nabla|^{\tau_2} b$, $A := I_{\tau_1-2K+\delta} |\nabla|^{\tau_1} a$, for some $\delta \in (\max\{2K - \tau_2, 0\}, 1 - s)$. Then Lemma 4.5 is applicable, and we have the claim.

We apply the same argument for (4.9). This concludes the proof of Lemma 4.2. \square

Proposition 4.3. *Let $f, g \in \mathcal{S}(\mathbb{R}^m)$, Then*

$$\| |\nabla|^\beta H_\mu(f, g) \|_{(p_0, 1), \mathbb{R}^m} \lesssim \| |\nabla|^\tau f \|_{\frac{m}{\kappa+\tau-\mu}, 2} \| |\nabla|^\mu g \|_2,$$

where τ is chosen as in Lemma 4.1

$$\frac{1}{p_0} = \frac{1}{2} + \frac{\kappa + \beta - \mu}{m}.$$

Proposition 4.4. *Let $f, g \in \mathcal{S}(\mathbb{R}^m)$, $\text{supp } f \subset \overline{B_r}$. Then for any $k \geq 2$,*

$$\| |\nabla|^\beta H_\mu(f, g) \|_{(p_0, 1), A_r^k} \lesssim 2^{k(-\frac{m}{2} + \kappa - \mu)} \| |\nabla|^\tau f \|_{\frac{m}{\kappa+\tau-\mu}} \| |\nabla|^\mu g \|_2,$$

where

$$\frac{1}{p_0} = \frac{1}{2} + \frac{\kappa + \beta - \mu}{m}.$$

Proof. Pick $\psi \in C_0^\infty(A_r^k)$, $\|\psi\|_{(p'_0, \infty)} \leq 1$, such that

$$\begin{aligned}
& \| |\nabla|^\beta H_\mu(g, f) \|_{(p_0, 1), A_{4r}^k} \\
& \lesssim \int \psi |\nabla|^\beta H_\mu(g, f) \\
& = \int |\nabla|^{\mu+\beta} \psi g f + \int |\nabla|^\beta \psi |\nabla|^\mu g f + \int |\nabla|^\beta \psi |\nabla|^\mu f g \\
& \lesssim \| |\nabla|^{\mu+\beta} \psi \|_{\infty, B_r} \|g f\|_1 + \|f\|_2 \| |\nabla|^\beta \psi \|_\infty \| |\nabla|^\mu g \|_2 + \| |\nabla|^{\mu-\tau} (g |\nabla|^\beta \psi) \|_{\frac{m}{m-\kappa-\tau+\mu}} \| |\nabla|^\tau f \|_{(\frac{m}{\kappa+\tau-\mu}, \infty)} \\
& \lesssim \| |\nabla|^{\mu+\beta} \psi \|_{\infty, B_r} \| |\nabla|^\mu g \|_2 \| |\nabla|^\tau f \|_{(\frac{m}{\kappa+\tau-\mu})} r^{\frac{m}{2}-\kappa+2\mu} \\
& \quad + \| |\nabla|^\beta \psi \|_{\infty, B_r} \| |\nabla|^\mu g \|_2 \| |\nabla|^\tau f \|_{(\frac{m}{\kappa+\tau-\mu})} r^{\frac{m}{2}-\kappa+\mu} \\
& \quad + \| |\nabla|^{\mu-\tau} (g |\nabla|^\beta \psi) \|_{\frac{m}{m-\kappa-\tau+\mu}} \| |\nabla|^\tau f \|_{(\frac{m}{\kappa+\tau-\mu})}
\end{aligned}$$

Lemma B.1 gives that

$$\begin{aligned}
\| |\nabla|^{\mu+\beta} \psi \|_{\infty, B_r} & \lesssim (2^k r)^{-m-\mu-\beta+\frac{m}{p'_0}} \|\psi\|_{p'_0} \lesssim (2^k r)^{\kappa-2\mu-\frac{m}{2}}, \\
\| |\nabla|^\beta \psi \|_{\infty, B_r} & \lesssim (2^k r)^{\kappa-\mu-\frac{m}{2}}.
\end{aligned}$$

Next, according to Lemma B.3

$$\begin{aligned}
& \| |\nabla|^{\mu-\tau} (g |\nabla|^\beta \psi) \|_{\frac{m}{m-\kappa-\tau+\mu}} \\
& \lesssim \sup_{\alpha \in [0, \mu-\tau]} (2^l r)^{-m-\beta-\alpha} r^\alpha \|\psi\|_1 \| |\nabla|^{\mu-\tau} g \|_{\frac{m}{m-\kappa-\tau+\mu}} \\
& \lesssim \sup_{\alpha \in [0, \mu-\tau]} (2^l r)^{-m-\beta-\alpha} r^\alpha (2^l r)^{\frac{m}{2}+\kappa+\beta-\mu} \|\psi\|_{p'_0, \infty} \| |\nabla|^\mu g \|_2 r^{-\kappa+\mu+\frac{m}{2}} \\
& = 2^{l(-\frac{m}{2}+\kappa-\mu)} \| |\nabla|^\mu g \|_2.
\end{aligned}$$

□

4.2. Pointwise commutator estimates via potentials

In this section, we discuss for commutators of which special cases have been appearing in [CRW76, Cha82]. There, usually estimates in the Hardy-space and BMO were proven. In contrast, we are going to prove pointwise estimates adapting our arguments from [Sch11], which might be of independent interest.

Lemma 4.5. *Let $\beta + \delta < \min(\tau, 1)$, $\delta > 0$, $\epsilon > 0$. There exists a finite number L , and $s_k, \tilde{s}_k > 0$, $t_k, \tilde{t}_k \in (0, \tau)$, $\tilde{s}_k + \tilde{t}_k = s_k + t_k = \tau - \beta - \delta$, $\tilde{s}_k < \epsilon$,*

$$\begin{aligned}
& \mathcal{C}(I_\tau A, |\nabla|^\beta)[|\nabla|^\delta B] \\
& \lesssim \sum_{k=1}^L I_{s_k} |A| I_{t_k} |B| + \sum_{k=1}^L I_{\tilde{s}_k} (I_{\tilde{t}_k} |A| |B|).
\end{aligned} \tag{4.10}$$

Proof. Since $\beta < 1$,

$$\begin{aligned}
|\nabla|^\beta (I_\tau A |\nabla|^\delta B)(x) & = \int \frac{I_\tau A(x) |\nabla|^\delta B(x) - I_\tau A(y) |\nabla|^\delta B(y)}{|x-y|^{m+\beta}} dy \\
& = \int \frac{I_\tau A(x) |\nabla|^\delta B(x) - I_\tau A(y) |\nabla|^\delta B(y)}{|x-y|^{m+\beta}} dy \\
& = I_\tau A(x) \int \frac{|\nabla|^\delta B(x) - |\nabla|^\delta B(y)}{|x-y|^{m+\beta}} dy \\
& \quad + \int \frac{(I_\tau A(x) - I_\tau A(y)) |\nabla|^\delta B(y)}{|x-y|^{m+\beta}} dy.
\end{aligned}$$

That is,

$$\begin{aligned}
& \mathcal{C}(I_\tau A, |\nabla|^\beta)[|\nabla|^\delta B](x) \\
&= \int \frac{(I_\tau A(x) - I_\tau A(y)) |\nabla|^\delta B(y)}{|x - y|^{m+\beta}} dy \\
&= \int \int \frac{(I_\tau A(x) - I_\tau A(y)) (B(y+w) - B(y))}{|w|^{m+\delta} |x - y|^{m+\beta}} dw dy \\
&= \int \int \left(\frac{(I_\tau A(x) - I_\tau A(y-w))}{|x - y + w|^{m+\beta}} - \frac{(I_\tau A(x) - I_\tau A(y))}{|x - y|^{m+\beta}} \right) B(y) \frac{dw}{|w|^{m+\delta}} dy \\
&= \int \int \int \left(\frac{(|x - z|^{-m+\tau} - |z - y + w|^{-m+\tau})}{|x - y + w|^{m+\beta}} - \frac{(|x - z|^{-m+\tau} - |z - y|^{-m+\tau})}{|x - y|^{m+\beta}} \right) B(y) A(z) \frac{dw}{|w|^{m+\delta}} dy dz.
\end{aligned}$$

So let us investigate the actual singularity of

$$k(x, y, z, w) := |w|^{-m-\delta} \left| \frac{(|x - z|^{-m+\tau} - |z - y + w|^{-m+\tau})}{|x - y + w|^{m+\beta}} - \frac{(|x - z|^{-m+\tau} - |z - y|^{-m+\tau})}{|x - y|^{m+\beta}} \right|.$$

We are going to show the following, for several sets $X \subset \mathbb{R}^{4m}$, which, as the union of these X is \mathbb{R}^{4m} , gives the claim.

$$\int \int \int \chi_X(x, z, y, w) k(x, z, y, w) A(z) B(y) dw dx dy \lesssim (4.10). \quad (4.11)$$

We are denoting k_1, k_2 ,

$$k(x, y, z, w) \leq k_1(x, y, z, w) + k_2(x, y, z, w), \quad (4.12)$$

where

$$\begin{aligned}
k_1(x, y, z, w) &:= |w|^{-m-\delta} \left| \frac{(|x - z|^{-m+\tau} - |z - y + w|^{-m+\tau})}{|x - y + w|^{m+\beta}} \right|, \\
k_2(x, y, z, w) &:= |w|^{-m-\delta} \left| \frac{(|x - z|^{-m+\tau} - |z - y|^{-m+\tau})}{|x - y|^{m+\beta}} \right|.
\end{aligned}$$

We split up the space $(x, y, z, w) \in \mathbb{R}^{4m}$ as follows

$$A_1 := \{(x, y, z, w) \in \mathbb{R}^{4m} : |x - y| \leq 2|z - y| \ \wedge \ |x - y| \leq 2|x - z|\},$$

$$A_2 := \{(x, y, z, w) \in \mathbb{R}^{4m} : |x - y| \leq 2|z - y| \ \wedge \ |x - y| > 2|x - z|\},$$

$$A_3 := \{(x, y, z, w) \in \mathbb{R}^{4m} : |x - y| > 2|z - y| \ \wedge \ |x - y| \leq 2|x - z|\},$$

$$B_1 := \{(x, y, z, w) \in \mathbb{R}^{4m} : |x - y + w| \leq 2|z - y + w| \ \wedge \ |x - y + w| \leq 2|x - z|\},$$

$$B_2 := \{(x, y, z, w) \in \mathbb{R}^{4m} : |x - y + w| \leq 2|z - y + w| \ \wedge \ |x - y + w| > 2|x - z|\},$$

$$B_3 := \{(x, y, z, w) \in \mathbb{R}^{4m} : |x - y + w| > 2|z - y + w| \ \wedge \ |x - y + w| \leq 2|x - z|\},$$

$$C_1 := \{(x, y, z, w) \in \mathbb{R}^{4m} : |w| \geq 4|x - y|\},$$

$$C_2 := \{(x, y, z, w) \in \mathbb{R}^{4m} : \frac{1}{4}|x - y| \leq |w| < 4|x - y|\},$$

$$C_3 := \{(x, y, z, w) \in \mathbb{R}^{4m} : 4|w| < |x - y|\}.$$

Note that

$$\bigcup_{i=1}^3 A_i = \bigcup_{i=1}^3 B_i = \bigcup_{i=1}^3 C_i = \mathbb{R}^{4m}.$$

Ad $(C_1 \cup C_2) \cap B_1$: First we observe that

$$\int \chi_{C_1 \cup C_2}(x, z, y, w) k_2(x, y, z, w) dw \lesssim \frac{||x - z|^{-m+\tau} - |z - y|^{-m+\tau}|}{|x - y|^{m+\beta+\delta}}.$$

Following the argument as in the proof of Lemma 4.2, we consequently have for a finite number L , some $s_k, \tilde{s}_k > 0$, $t_k, \tilde{t}_k \in (0, \tau)$, $\tilde{s}_k + \tilde{t}_k = s_k + t_k = \tau - \beta - \delta$, $\tilde{s}_k < \epsilon$, such that

$$\int \int \int \chi_{C_1 \cup C_2}(x, z, y, w) k_2(x, z, y, w) A(z) B(y) dw dx dy \lesssim \sum_{k=1}^L I_{s_k} |A|(x) I_{t_k} |B|(x) + \sum_{k=1}^L I_{\tilde{s}_k} (I_{\tilde{t}_k} |A| |B|)(x). \quad (4.13)$$

Moreover, for any $\epsilon \in [0, 1]$, since on B_1 , $|x - z| \approx |z - y + w|$, we can use the mean value theorem and have

$$\chi_{(C_1 \cup C_2) \cap B_1} k_1(x, z, y, w) \lesssim |w|^{-m-\delta} \max\{|x - z|, |z - y + w|\}^{-m+\tau-\epsilon} |x - y + w|^{-m-\beta+\epsilon} \chi_{(C_1 \cup C_2) \cap B_1}$$

Now,

$$\begin{aligned} |x - y + w| \chi_{C_1} &\gtrsim |w| \chi_{C_1} \gtrsim |x - y| \chi_{C_1}, \\ |x - y + w| \chi_{C_2} &\lesssim |w| \chi_{C_2} \approx |x - y| \chi_{C_2}, \end{aligned}$$

Consequently,

$$\begin{aligned} \chi_{(C_1 \cup C_2) \cap B_1} k_1(x, z, y, w) &\lesssim |w|^{-m-\delta} \chi_{|w| \gtrsim |x-y|} |x - z|^{-m+\tau-\epsilon} |x - y|^{-m-\beta+\epsilon} \\ &\quad + |x - y|^{-m-\delta} |x - z|^{-m+\tau-\epsilon} |x - y + w|^{-m-\beta+\epsilon} \chi_{|x-y+w| \lesssim |x-y|} \end{aligned}$$

Integrating in w implies then for any $\delta > 0$, $\epsilon \in (\beta, 1)$

$$\int \chi_{(C_1 \cup C_2) \cap B_1} k_1(x, z, y, w) dw \lesssim |x - y|^{-m-\delta-\beta+\epsilon} |x - z|^{-m+\tau-\epsilon},$$

thus if we choose $\epsilon \in (\delta + \beta, \tau)$, $s := \tau - \epsilon$, $t := -\delta - \beta + \epsilon$, together with (4.13), we have shown (4.11) for $X = (C_1 \cup C_2) \cap B_1$.

Ad $(C_1 \cup C_2) \cap B_2$: Next, we consider $\chi_{(C_1 \cup C_2) \cap B_2} k$:

$$\chi_{(C_1 \cup C_2) \cap B_2} k_1(x, y, z, w) \lesssim |w|^{-m-\delta} \frac{|x - z|^{-m+\tau}}{|x - y + w|^{m+\beta}} \chi_{(C_1 \cup C_2) \cap B_2} \lesssim |w|^{-m-\delta} \frac{|x - z|^{-m+\tau-\epsilon}}{|x - y + w|^{m+\beta-\epsilon}} \chi_{(C_1 \cup C_2) \cap B_2},$$

Now one proceeds exactly as in the situation for B_1 above and we have (4.11) for $X = (C_1 \cup C_2) \cap (B_1 \cup B_2)$.

Ad $(C_1 \cup C_2) \cap B_3$: Then, we have to consider $\chi_{(C_1 \cup C_2) \cap B_3} k$

$$\chi_{(C_1 \cup C_2) \cap B_3} k_1(x, y, z, w) \lesssim |w|^{-m-\delta} \frac{|z-y+w|^{-m+\tau-\varepsilon}}{|x-y+w|^{m+\beta-\varepsilon}} \chi_{(C_1 \cup C_2) \cap B_3},$$

Using that

$$\chi_{(C_1 \cup C_2)} |w| \gtrsim \max\{|x-y+w|, |x-y|\} \chi_{(C_1 \cup C_2)},$$

for any $\varepsilon_1 + \varepsilon_2 = \varepsilon < \tau$

$$\begin{aligned} \int \chi_{(C_1 \cup C_2) \cap B_3} k_1(x, y, z, w) A(z) dz &\lesssim |x-y|^{-m-\delta+\varepsilon_1} |x-y+w|^{-m-\beta+\varepsilon_2} \int |z-y+w|^{-m+\tau-\varepsilon} A(z) dz \\ &\approx |x-y|^{-m-\delta+\varepsilon_1} |x-y+w|^{-m-\beta+\varepsilon_2} I_{\tau-\varepsilon} A(y-w), \end{aligned}$$

that is for any $\varepsilon_2 > \beta$,

$$\begin{aligned} \int \int \chi_{(C_1 \cup C_2) \cap B_3} k_1(x, y, z, w) A(z) dz dw &\lesssim |x-y|^{-m-\delta+\varepsilon_1} \int |x-y+w|^{-m-\beta+\varepsilon_2} I_{\tau-\varepsilon} A(y-w) dw \\ &\approx |x-y|^{-m-\delta+\varepsilon_1} \int |x-\tilde{w}|^{-m-\beta+\varepsilon_2} I_{\tau-\varepsilon} A(\tilde{w}) d\tilde{w} \\ &\approx |x-y|^{-m-\delta+\varepsilon_1} I_{\tau-\beta-\varepsilon_1} A(x) \end{aligned}$$

which gives for $\varepsilon_2 > \beta$ the claim (4.11) for $X = (C_1 \cup C_2) \cap B_3$, where $s = \tau - \beta - \varepsilon_1$ and $t = \varepsilon_1 - \delta$. Together, (4.11) holds for $X \subseteq C_1 \cup C_2$.

Ad C_3 : It remains to show the claim for C_3 :

$$\chi_{C_3}(x, y, z, w) |x-y| \approx \chi_{C_3}(x, y, z, w) |x-y+w|$$

$$\chi_{C_3 \cap (A_1 \cup A_2)}(x, y, z, w) |z-y| \approx \chi_{C_3 \cap (A_1 \cup A_2)}(x, y, z, w) |z-y+w|$$

In this case, k_1 and k_2 should not be considered independently, but we rather use the following

$$\begin{aligned} k(x, y, z, w) &\leq |w|^{-m-\delta} \frac{\left| |x-z|^{-m+\tau} - |z-y|^{-m+\tau} \right|}{|x-y|^{m+\beta} |x-y+w|^{m+\beta}} \left| |x-y|^{m+\beta} - |x-y+w|^{m+\beta} \right| \\ &\quad + |w|^{-m-\delta} \frac{\left| |z-y|^{-m+\tau} - |z-y+w|^{-m+\tau} \right|}{|x-y+w|^{m+\beta}}. \end{aligned} \tag{4.14}$$

Note that

$$\chi_{C_3} |x-y+w| \gtrsim \chi_{C_3} |x-y|.$$

$$\chi_{C_3 \cap (A_1 \cup A_2)} |y-z+w| \approx \chi_{C_3 \cap (A_1 \cup A_2)} |z-y|.$$

Thus, using (4.14) with the mean value formula or the conditions A_2 for any $\varepsilon = \varepsilon_1 + \varepsilon_2 \in [0, 1]$

$$\begin{aligned} \chi_{C_3 \cap (A_1 \cup A_2)} k(x, y, z, w) &\leq |w|^{-m-\delta} \frac{\min\{|x-z|, |z-y|\}^{-m+\tau-\varepsilon} |x-y|^\varepsilon}{|x-y|^{2m+2\beta}} |x-y|^{m+\beta-\varepsilon} |w|^\varepsilon \chi_{C_3 \cap (A_1 \cup A_2)} \\ &\quad + |w|^{-m-\delta} \frac{1}{|x-y|^{m+\beta}} |z-y|^{-m+\tau-\varepsilon} |w|^\varepsilon \chi_{C_3 \cap (A_1 \cup A_2)} \\ &\lesssim |w|^{-m-\delta} \frac{|x-z|^{-m+\tau-\varepsilon}}{|x-y|^{m+\beta}} |w|^{\varepsilon_1} |x-y|^{\varepsilon_2} \chi_{C_3 \cap (A_1 \cup A_2)} \\ &\quad + |w|^{-m-\delta} \frac{1}{|x-y|^{m+\beta}} |z-y|^{-m+\tau-\varepsilon} |w|^{\varepsilon_1} |x-y|^{\varepsilon_2} \chi_{C_3 \cap (A_1 \cup A_2)} \end{aligned}$$

Consequently, if we choose $\varepsilon_1 \in (\delta, \delta + \epsilon/2)$, $\varepsilon_2 \in (\beta, \beta + \epsilon/2)$ such that $\varepsilon = \varepsilon_1 + \varepsilon_2 < \min\{1, \tau\}$, using that

$$\int \chi_{C_3} |w|^{-m-\delta+\varepsilon_1} dw \approx |x-y|^{\varepsilon_1-\delta},$$

we arrive at

$$\begin{aligned} \int \chi_{C_3 \cap (A_1 \cup A_2)} k(x, y, z, w) dw &\lesssim |x-z|^{-m+\tau-\varepsilon} |x-y|^{-m+\varepsilon-\delta-\beta} \\ &\quad + |z-y|^{-m+\tau-\varepsilon} |x-y|^{-m-\varepsilon-\delta-\beta}. \end{aligned}$$

This implies for $s := \tau - \varepsilon > 0$, $t := \varepsilon - \delta - \beta \in (0, \epsilon)$, (4.11) for $X = C_3 \cap (A_1 \cup A_2)$:

$$\int \int \int \chi_{C_3 \cap (A_1 \cup A_2)} k(x, y, z, w) B(y) A(z) dw dy dz \lesssim I_s |A|(x) I_t |B|(x) + I_t (|B| I_s |A|)(x).$$

Ad $C_3 \cap A_3$: The last case is $C_3 \cap A_3$, where we have by (4.14)

$$\begin{aligned} \chi_{C_3 \cap A_3} k(x, y, z, w) &\lesssim |w|^{-m-\delta+\varepsilon} \frac{|z-y|^{-m+\tau}}{|x-y|^{m+\beta}} |x-y|^{-\varepsilon} \chi_{C_3 \cap A_3} \\ &\quad + |w|^{-m-\delta} \frac{\left| |z-y|^{-m+\tau} - |z-y+w|^{-m+\tau} \right|}{|x-y|^{m+\beta}} \chi_{C_3 \cap A_3} \\ &\lesssim |w|^{-m-\delta+\varepsilon_1} \frac{|z-y|^{-m+\tau-\varepsilon}}{|x-y|^{m+\beta-\varepsilon_2}} \chi_{C_3 \cap A_3} \\ &\quad + |w|^{-m-\delta-\varepsilon_2} \frac{\left| |z-y|^{-m+\tau} - |z-y+w|^{-m+\tau} \right|}{|x-y|^{m+\beta-\varepsilon_2}} \chi_{C_3 \cap A_3} \end{aligned}$$

While the first term on the right-hand side behaves exactly as before, for the second term we need yet another case study: If $|w| \leq \frac{1}{2}|z-y|$, we can proceed as in the cases before using the mean value formula. Note that on the other hand,

$$\chi_{|w| > \frac{1}{2}|z-y|} |z-y+w| < 3|w|,$$

Consequently,

$$\begin{aligned} &\chi_{|w| > \frac{1}{2}|z-y|} |w|^{-m-\delta-\varepsilon_2} \frac{\left| |z-y|^{-m+\tau} - |z-y+w|^{-m+\tau} \right|}{|x-y|^{m+\beta-\varepsilon_2}} \chi_{C_3 \cap A_3} \\ &\lesssim \chi_{|z-y| < 2|w|} |w|^{-m-\delta-\varepsilon_2} \frac{|z-y|^{-m+\tau}}{|x-y|^{m+\beta-\varepsilon_2}} \chi_{C_3 \cap A_3} \\ &\quad + \chi_{|z-y+w| < 3|w|} |w|^{-m-\delta-\varepsilon_2+\varepsilon} \frac{|z-y+w|^{-m+\tau-\varepsilon}}{|x-y|^{m+\beta-\varepsilon_2}} \chi_{C_3 \cap A_3} \\ &=: I + II. \end{aligned}$$

Since

$$\int \chi_{|w| > \frac{1}{2}|z-y|} |w|^{-m-\delta-\varepsilon_2} dw \approx |z-y|^{-\delta-\varepsilon_2},$$

we arrive at

$$\int I dw = \frac{|z-y|^{-m+\tau-\delta-\varepsilon_2}}{|x-y|^{m+\beta-\varepsilon_2}} \chi_{C_3 \cap A_3}$$

so setting $\varepsilon = \varepsilon_1 + \varepsilon_2 \in (\beta + \delta, \min\{1, \tau, \beta + \delta + \epsilon\})$, for $\varepsilon_1 \in (\delta, \delta + \epsilon/2)$, $\varepsilon_2 \in (\beta, \beta + \epsilon/2)$ gives the estimate (4.11) for I , and as for II ,

$$\int |w|^{-m-\delta-\varepsilon_2+\varepsilon} \frac{|z-y+w|^{-m+\tau-\varepsilon}}{|x-y|^{m+\beta-\varepsilon_2}} |A|(z) dz \stackrel{\varepsilon \leq \tau}{\approx} |w|^{-m-\delta-\varepsilon_1} |x-y|^{-m-\beta+\varepsilon_2} I_{\tau-\varepsilon} |A|(y-w),$$

$$\int |w|^{-m-\delta-\varepsilon_2+\varepsilon} |x-y|^{-m-\beta+\varepsilon_2} I_{\tau-\varepsilon}|A|(y-w) dw \stackrel{\varepsilon_1>\delta}{\approx} |x-y|^{-m-\beta+\varepsilon_2} I_{\tau-\delta-\varepsilon_2}|A|(y),$$

and finally,

$$\int |x-y|^{-m-\beta+\varepsilon_2} (I_{\tau-\delta-\varepsilon_2}|A|(y)) |B|(y) dy \stackrel{\varepsilon_2>\beta}{\approx} I_{\varepsilon_2-\beta}((I_{\tau-\delta-\varepsilon_2}|A|) |B|)(x).$$

Thus, also II has the required estimate and we have shown (4.11) for $X = C_3 \cap A_3$. \square

The following estimate should be compared to the estimates in [Cha82], who extended arguments in [CRW76] from Riesz transforms to Riesz Potentials. Their estimates treat cases in which one of the involved functions b belongs to BMO , which one usually uses in applications for estimates of that expression in terms of $|\nabla|^s b$. But if one knows that $|\nabla|^s b$ exists, then the following estimates are more precise than their BMO -counterparts in terms of Lorentz space estimates.

Lemma 4.6. *For any $\delta > 0$ such that $s + \delta < 1$ and any $\gamma \in (s, s + \delta)$, we have*

$$\begin{aligned} \|\nabla|^s \mathcal{C}(a, I_s)[b]\| &\leq C_{s,\delta,\gamma} I_{s+\delta-\gamma} \left| |\nabla|^{s+\delta} a \right| I_{\gamma-s} |I_s b| \\ &\quad + C_{s,\delta,\gamma} \min \left\{ I_{\gamma-s} \left(|I_s b| I_{s+\delta-\gamma} \left| |\nabla|^{s+\delta} a \right| \right), I_{\gamma-s} \left(I_{s+\delta-\gamma} |I_s b| \left| |\nabla|^{s+\delta} a \right| \right) \right\} \end{aligned}$$

Proof. For $\delta > 0$ such that $s + \delta < 1$. Set

$$B := I_s b, \quad A := |\nabla|^{s+\delta} a.$$

Then,

$$\mathcal{C}(a, I_s)[b] = I_s((I_{s+\delta}A)(|\nabla|^s B) - |\nabla|^s((I_{s+\delta}A)B)).$$

Now,

$$\begin{aligned} |\nabla|^s((I_{s+\delta}A)B)(x) &= c_s \int_{\mathbb{R}^m} \frac{I_{s+\delta}A(x) B(x) - I_{s+\delta}A(y) B(y)}{|x-y|^{n+s}} dy \\ &= I_{s+\delta}A(x) c_s \int_{\mathbb{R}^m} \frac{B(x) - B(y)}{|x-y|^{n+s}} dy + c_s \int_{\mathbb{R}^m} \frac{I_{s+\delta}A(x) - I_{s+\delta}A(y)}{|x-y|^{n+s}} B(y) dy \\ &= I_{s+\delta}A(x) |\nabla|^s B(x) + c_{s,\delta} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{|z-x|^{-n+(s+\delta)} - |z-y|^{-n+(s+\delta)}}{|x-y|^{n+s}} A(z) B(y) dz dy \end{aligned}$$

Let now $\gamma \in (s, s + \delta) \subset (0, 1)$ and denote

$$k(x, y, z) := \frac{|z-x|^{-n+(s+\delta)} - |z-y|^{-n+(s+\delta)}}{|x-y|^{n+s}}.$$

Now we follow the strategy in [Sch11]. We decompose the space $(x, y, z) \in \mathbb{R}^{3n}$ into several subspaces depending on the relations of $|z-y|$, $|x-y|$, $|x-z|$:

$$1 \leq \chi_1(x, y, z) + \chi_2(x, y, z) + \chi_3(x, y, z) + \chi_4(x, y, z) \quad \text{for } x, y, z \in \mathbb{R}^m,$$

where

$$\chi_1 := \chi_{|x-y| \leq 2|z-y|} \chi_{|x-y| \leq 2|x-z|},$$

$$\chi_2 := \chi_{|x-y| \leq 2|z-y|} \chi_{|x-y| > 2|x-z|},$$

$$\chi_3 := \chi_{|x-y| > 2|z-y|} \chi_{|x-y| \leq 2|x-z|} \chi_{|x-z| \leq 2|z-y|},$$

$$\chi_4 := \chi_{|x-y| > 2|z-y|} \chi_{|x-y| \leq 2|x-z|} \chi_{|x-z| > 2|z-y|}.$$

Then, by the mean value theorem

$$\chi_1(x, y, z)k(x, y, z) \lesssim |z-x|^{-n+s+\delta-\gamma} |y-x|^{-n-s+\gamma}.$$

Same holds for

$$\chi_2(x, y, z)k(x, y, z) + \chi_3(x, y, z)k(x, y, z) \lesssim |z - x|^{-n+s+\delta-\gamma} |y - x|^{-n-s+\gamma}.$$

Finally,

$$\chi_4(x, y, z)k(x, y, z) \lesssim |z - y|^{-n+s+\delta-\gamma} |y - x|^{-n-s+\gamma}.$$

Note that

$$\chi_4(x, y, z)|x - y| \leq \chi_4(x, y, z)|x - z| + \chi_4(x, y, z)|z - y| \leq \frac{3}{2}\chi_4(x, y, z)|x - z|,$$

and

$$\chi_4(x, y, z)|x - y| \geq \chi_4(x, y, z)|x - z| - \chi_4(x, y, z)|z - y| \geq \frac{1}{2}\chi_4(x, y, z)|x - z|,$$

that is

$$\chi_4(x, y, z)|x - y| \approx \chi_4(x, y, z)|x - z|.$$

That is,

$$\chi_4(x, y, z)k(x, y, z) \lesssim |z - y|^{-n+s+\delta-\gamma} \min\{|y - x|^{-n-s+\gamma}, |z - x|^{-n-s+\gamma}\}$$

This implies,

$$\begin{aligned} & \|\nabla|^s((I_{s+\delta}A)B)(x) - I_{s+\delta}A(x) \nabla|^s B(x)\| \\ & \leq c_{s,\delta,\gamma} I_{s+\delta-\gamma}|A| I_{\gamma-s}|B| + c_{s,\delta,\gamma} \min\{I_{\gamma-s}(|B| I_{s+\delta-\gamma}|A|), I_{\gamma-s}(I_{s+\delta-\gamma}|B| |A|)\}. \end{aligned}$$

It is a simple adaption, to show that in the claim, for both terms on the right-hand side, we could have chosen different γ . \square

For $s = 0$, a (non-trivial) version of Lemma 4.6, is the following result, for any Riesz transform \mathcal{R} . Like Lemma 4.6 was related to Chanillo's [Cha82], this estimate is related to [CRW76]. The proof follows by the same arguments as the proof of Lemma 4.6, we leave the details to the reader.

Lemma 4.7. *Then, for any $\delta \in (0, 1)$ and any $\gamma_i \in (0, \delta)$, $i = 1, 2$, we have*

$$|\mathcal{C}(a, \mathcal{R})[b]| \leq C_{\mathcal{R},\delta,\gamma_1} I_{\delta-\gamma_1} \left| |\nabla|^\delta a \right| I_{\gamma_1} |b| + C_{\mathcal{R},\delta,\gamma_2} I_{\gamma_2} \left(I_{\delta-\gamma_2} |b| \left| |\nabla|^\delta a \right| \right).$$

4.3. Fractional product rules in the Hardy-space via para-products – including the limit case

In this section we introduce several commutators, and show how to use techniques developed by Da Lio and Rivière in [DLR11b] in order to estimate their behavior involving the Hardy spaces \mathcal{H} . For the case $\mu < 1$ the new contribution are the commutators themselves, the techniques for the proof of their behavior follows the arguments of Da Lio and Rivière. In the case $\mu = 1$, these arguments have to be extended and more precise, in order to show the same behavior for this limit case, which was unknown up to now. The essential commutator estimate is

$$\| |\nabla|^\mu (\mathcal{R}[h] I_\mu b - \mathcal{R}[h I_\mu b]) \|_{\mathcal{H}} \leq \|h\|_2 \|b\|_2 \quad \text{whenever } \mu \in (0, 1]. \quad (4.15)$$

From this we will conclude

$$\|\mathcal{C}(f, \mathcal{R})[|\nabla|^\mu \varphi]\|_2 \lesssim \| |\nabla|^\mu f \|_2 \|\varphi\|_{BMO}, \quad (4.16)$$

and

$$\|H_\mu(\varphi, g)\|_2 \lesssim \| |\nabla|^\mu g \|_2 \|\varphi\|_{BMO}. \quad (4.17)$$

as well as

$$\| |\nabla|^\mu H_\mu(a, b) \|_{\mathcal{H}} \lesssim \| |\nabla|^\mu a \|_2 \| |\nabla|^\mu b \|_2 \quad \text{for } \mu \in (0, 1]. \quad (4.18)$$

Note that (4.18) is already known for $\mu < 1$ [DLR11b], and we are going to prove it only for the new situation $\mu = 1$, when we will show that it follows from (4.15).

Proof of (4.15). Let

$$T(h, b) := |\nabla|^\mu (\mathcal{R}[h] I_\mu b - \mathcal{R}[h I_\mu b])$$

We will follow the basic ideas of the proof of the commutator theorem in [DLR11b], which goes through without great changes, if $\mu < 1$, and then point out where it fails if $\mu = 1$. In the latter case, we have make a precise computation of the failing term, and show that in fact this term is essentially a commutator as in [CRW76] and another good term. For a general reference, we refer the reader to the proof in [DLR11b], as we will sketch some of the parts which behave no worse than in their case. In particular, the notion of Besov- and Triebel spaces and their respective estimates are taken from [DLR11b] without too much changes.

The para-products and Littlewood-Paley decomposition

We need to control three parts of T . We introduce the projections Π_1, Π_2, Π_3 defined via

$$\begin{aligned} \Pi_1 T(f, g) &:= \sum_{j \in \mathbb{Z}} T(f_j, g^{j-4}), \\ \Pi_2 T(f, g) &:= \sum_{j \in \mathbb{Z}} T(f^{j-4}, g_j), \\ \Pi_3 T(f, g) &:= \sum_{j \in \mathbb{Z}} \sum_{l=j-4}^{j+4} T(f_j, g_l). \end{aligned}$$

Here,

$$\text{supp } f_j^\wedge \subset \{|\xi| \in (2^{j-1}, 2^{j+1})\},$$

and

$$\text{supp}(f^j)^\wedge \subset \{|\xi| \leq 2^{j+1}\}.$$

These terms come from the Littlewood-Paley decomposition. Note that then for $k \in \mathbb{Z}$,

$$\Pi_1 T(f, g)_k := \sum_{j=k-3}^{k+3} T(f_j, g^{j-4})_k, \quad (4.19)$$

$$\Pi_2 T(f, g)_k := \sum_{j=k-3}^{k+3} T(f^{j-4}, g_j)_k, \quad (4.20)$$

$$\Pi_3 T(f, g)_k := \sum_{j=k-5}^{\infty} \sum_{l=j-4}^{j+4} T(f_j, g_l)_k. \quad (4.21)$$

Estimates from [DLR11b]

Let us recall the following estimates, whose proof up to small adaptations can be found in [DLR11b], for a general overview we refer to Tao's lecture notes [Tao01]: First, denoting with \mathcal{M} the maximal function,

$$\sup_{k \in \mathbb{Z}} |2^{-\gamma k} |\nabla|^\gamma f^k(x)| \lesssim \mathcal{M}f(x) \quad \text{for any } \gamma \geq 0 \text{ and almost every } x \in \mathbb{R}^m. \quad (4.22)$$

Here, for $\gamma = 0$, we set $|\nabla|^0 := \text{Id}$. In fact, for a suitable $\tilde{\phi}$ in the Schwartz class,

$$2^{-\gamma k} |\nabla|^\gamma f^k(x) = 2^{(m-\gamma)k} \int \tilde{\phi}(2^k(x-y)) |\nabla|^\gamma f(y) = 2^{(m)j} \int (|\nabla|^\gamma \tilde{\phi})(2^j(x-y)) f(y).$$

Now the argument follows the exactly the one of [DLR11b], using that

$$\left| |\nabla|^\gamma \tilde{\phi}(z) \right| \lesssim \begin{cases} \frac{1}{1+|z|^{m+\gamma}} & \text{if } \gamma > 0 \\ \frac{1}{1+|z|^\beta} & \text{if } \gamma = 0, \text{ for any } \beta > 0, \end{cases}$$

and we arrive at

$$2^{-\gamma k} |\nabla|^\gamma f^k(x) \lesssim \mathcal{M}f(x) \sum_{j \in \mathbb{Z}} 2^{mj} \sup_{|z| \leq 2^j} \left| |\nabla|^\gamma \tilde{\phi}(z) \right|,$$

which is convergent if $\gamma \geq 0$, but might be divergent for $\gamma < 0$. This shows (4.22). \square

Let N be a zero-multiplier operator with 0-homogeneous symbol n . We have (cf. [Tao01, Notes 2 for 25A, Lemma 1.1])

$$\sup_k \|2^{-\gamma k} N |\nabla|^\gamma f_k\|_\infty \lesssim C_N \|f\|_{\dot{B}_{\infty,\infty}^0} \quad \text{for } \gamma \geq 0 \quad (4.23)$$

and thus also

$$\| |\nabla|^\gamma N f^j \|_\infty \lesssim 2^{\gamma j} \|f\|_{\dot{B}_{\infty,\infty}^0} \quad \text{for } \gamma > 0 \quad (4.24)$$

A crucial role is played by

$$\left(\int_{\mathbb{R}^m} \sum_{j \in \mathbb{Z}} \left| 2^{j(\gamma-\alpha)} N |\nabla|^{\alpha-\gamma} f^j \right|^2 \right)^{\frac{1}{2}} \lesssim C_{\alpha-\gamma, N} \|f\|_2 \quad \text{whenever } 0 \leq \gamma < \alpha. \quad (4.25)$$

which follows from

$$\left(\int_{\mathbb{R}^m} \sum_{j \in \mathbb{Z}} \left| 2^{j(\gamma-\alpha)} N |\nabla|^{\alpha-\gamma} f^j \right|^2 \right)^{\frac{1}{2}} \stackrel{\gamma < \alpha}{\lesssim} C_{\alpha-\gamma, N} \left(\int_{\mathbb{R}^m} \sum_{j \in \mathbb{Z}} \left| 2^{j(\gamma-\alpha)} |\nabla|^{\alpha-\gamma} f^j \right|^2 \right)^{\frac{1}{2}},$$

an argument which can be found in the commutator estimates in [DLR11b].

Estimate of Π_2

We have

$$\begin{aligned} \|\Pi_2 T(h, b)\|_{\mathcal{H}} &\lesssim \| |\nabla|^\mu \Pi_2(\mathcal{R}[h] I_\mu b)\|_{\mathcal{H}} + \|\mathcal{R}[|\nabla|^\mu \Pi_2(h I_\mu b)]\|_{\mathcal{H}} \\ &\lesssim \| |\nabla|^\mu \Pi_2(\mathcal{R}[h] I_\mu b)\|_{\mathcal{H}} + \| |\nabla|^\mu \Pi_2(h I_\mu b)\|_{\mathcal{H}}. \end{aligned}$$

Let $\tilde{h} := \mathcal{R}[h]$ or h , respectively. Then

$$\begin{aligned} \| |\nabla|^\mu \Pi_2(\tilde{h} I_\mu b)\|_{\mathcal{H}} &\approx \| |\nabla|^\mu \Pi_2(\tilde{h} I_\mu b)\|_{\dot{F}_{1,2}^0} \\ &\approx \|\Pi_2(\tilde{h} I_\mu b)\|_{\dot{F}_{1,2}^\mu} \\ &\approx \int_{\mathbb{R}^m} \left(\sum_{k \in \mathbb{Z}} 2^{2\mu k} |\Pi_2(\tilde{h} I_\mu b)|^2 \right)^{\frac{1}{2}} \\ &\stackrel{(4.20)}{\approx} \int_{\mathbb{R}^m} \left(\sum_{k \in \mathbb{Z}} 2^{2\mu k} \left| \sum_{j=k-3}^{k+3} (\tilde{h}^{j-4} I_\mu b_j)_k \right|^2 \right)^{\frac{1}{2}} \\ &\lesssim \int_{\mathbb{R}^m} \left(\sum_{k \in \mathbb{Z}} \max_{j \in [k-3, k+3] \cap \mathbb{Z}} 2^{2\mu k} |(\tilde{h}^{j-4} I_\mu b_j)_k|^2 \right)^{\frac{1}{2}} \\ &\lesssim \int_{\mathbb{R}^m} \left(\sum_{k \in \mathbb{Z}} 2^{2\mu k} (\sup_i |\tilde{h}^i|^2) |I_\mu b_k|^2 \right)^{\frac{1}{2}} \end{aligned}$$

Thus,

$$\begin{aligned} \| |\nabla|^\mu \Pi_2(\tilde{h} I_\mu b)\|_{\mathcal{H}} &\stackrel{(4.22)}{\lesssim} \int_{\mathbb{R}^m} \mathcal{M} \tilde{h} \left(\sum_{k \in \mathbb{Z}} 2^{2\mu k} |I_\mu b_k|^2 \right)^{\frac{1}{2}} \\ &\lesssim \|\tilde{h}\|_2 \|b\|_{\dot{B}_{2,2}^0} \\ &\lesssim \|h\|_2 \|b\|_2. \end{aligned}$$

It is worth noting, that this argument holds for any $\mu > 0$.

Estimate of Π_1

Now we treat $\Pi_1 T$. For some ψ , $\|\psi\|_{\dot{B}_{\infty,\infty}^0} \leq 1$ we have

$$\begin{aligned} \|\Pi_1 T(h, b)\|_{\mathcal{H}} &\lesssim \|\Pi_1 T(h, b)\|_{\dot{B}_{1,1}^0} \\ &\lesssim \int \Pi_1 T(h, b)^\wedge(\xi) \psi^\vee(\xi) d\xi \\ &= c \sum_{j \in \mathbb{Z}} \int \int k(\xi, \eta) h_j^\wedge(\eta) (b^{j-4})^\wedge(\xi - \eta) \psi^\vee(\xi) d\eta d\xi, \end{aligned}$$

where

$$k(\xi, \eta) = \left(\frac{|\xi|}{|\xi - \eta|} \right)^\mu \left(\frac{\eta}{|\eta|} - \frac{\xi}{|\xi|} \right). \quad (4.26)$$

Note that by the support of the different factors, we only consider ξ, η such that $|\xi - \eta| \lesssim 2^{j-3}$, $2^{j-1} \leq |\eta| \leq 2^{j+1}$, and thus $2^{j-2} \leq |\xi| \leq 2^{j+2}$. In particular,

$$\frac{|\xi - \eta|}{|\eta|} \leq \frac{1}{2}. \quad (4.27)$$

We can assume w.l.o.g. that if (4.27) is satisfied, following identity holds, and the right hand side converges absolutely

$$\begin{aligned} k(\xi, \eta) &= \left(\frac{|\xi|}{|\xi - \eta|} \right)^\mu \sum_{l=1}^{\infty} \frac{1}{l!} m_l(\eta) n_l(\xi - \eta) \left(\frac{|\xi - \eta|}{|\eta|} \right)^l \\ &= \sum_{l=1}^{\infty} \frac{1}{l!} m_l(\eta) n_l(\xi - \eta) |\xi - \eta|^{l-\mu} |\xi|^\mu |\eta|^{-l}. \end{aligned} \quad (4.28)$$

where n_l, m_l are zero-homogeneous functions, and we will denote their respective operators with N_l, M_l . In fact, one can check that this is true, if (4.27) is satisfied for 2^{-L} on the right-hand side, for some $L \in \mathbb{N}$. If L is not 1, then we need to replace for the decompositions Π_1, Π_2, Π_3 and use instead

$$\begin{aligned} \tilde{\Pi}_1 T(f, g) &:= \sum_{j \in \mathbb{Z}} T(f_j, g^{j-3-L}), \\ \tilde{\Pi}_2 T(f, g) &:= \sum_{j \in \mathbb{Z}} T(f^{j-3-L}, g_j), \\ \tilde{\Pi}_3 T(f, g) &:= \sum_{j \in \mathbb{Z}} \sum_{l=j-3-L}^{j+3+L} T(f_j, g_l). \end{aligned}$$

Thus, for the sake of simplicity, we are going to assume that $L = 1$.

Estimates for Π_1 if $\mu < 1$.

The expansion in (4.28) is precise enough, if $\mu < 1$, since then $l - \mu > 0$ for all $l \in \mathbb{N}$. Here $(\cdot)_{\bar{k}}$ is the Fourier cutoff on $|\xi| \in (2^{k-2}, 2^{k+2})$.

$$\begin{aligned} \|\Pi_1 T(h, b)\|_{\mathcal{H}} &\lesssim \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{j \in \mathbb{Z}} \int M_l I_l h_j N_l |\nabla|^{l-\mu} b^{j-4} |\nabla|^\mu \psi_{\bar{j}} \\ &\lesssim \sum_{l=1}^{\infty} \frac{1}{l!} \int \sum_{j \in \mathbb{Z}} |2^{jl} M_l I_l h_j| \left| 2^{j(\mu-l)} N_l |\nabla|^{l-\mu} b^{j-4} \right| \sup_{\bar{k}} \|2^{-\mu \bar{k}} |\nabla|^\mu \psi_{\bar{k}}\|_{\infty} \\ &\stackrel{(4.23)}{\lesssim} \sum_{l=1}^{\infty} \frac{1}{l!} 2^{4(\mu-l)} \left(\int_{\mathbb{R}^m} \sum_{j \in \mathbb{Z}} |2^{jl} M_l I_l h_j|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^m} \sum_{j \in \mathbb{Z}} |2^{(j-4)(\mu-l)} N_l |\nabla|^{l-\mu} b^{j-4}|^2 \right)^{\frac{1}{2}} \quad (4.29) \\ &\stackrel{l \geq 1 > \mu}{\stackrel{(4.25)}}{\lesssim} C_{1-\mu} \sum_{l=1}^{\infty} c_l 2^{4(\mu-l)} \|h\|_2 \|b\|_2 \\ &\lesssim \|h\|_2 \|b\|_2. \end{aligned}$$

The crucial point in the above estimate is the application of (4.25), and it was there that we used that $\mu < l$ for all $l \geq 1$, that is $\mu < 1$. In particular, if the summation on l above had started with $l = 2$, this estimate would have been true for any $\mu < 2$. Also, note that for $\mu = 0$, this estimate still holds.

Adaptions for Π_1 if $\mu = 1$.

Thus, if $\mu = 1$, we have to be more precise and actually need to find good controls of the Taylor-expansion for $l = 1$:

$$k(\xi, \eta) \stackrel{(4.26)}{=} -k_1(\xi, \eta) + \sum_{l=2}^{\infty} \frac{1}{l!} m_l(\eta) n_l(\xi - \eta) |\xi - \eta|^{l-1} |\eta|^{1-l},$$

where

$$\begin{aligned} k_1(\xi, \eta) &= \frac{|\xi|}{|\xi - \eta|} \frac{-(\xi - \eta)|\eta| + \eta \frac{\eta^k}{|\eta|} (\xi - \eta)^k}{|\eta|^2} \\ &= \frac{|\xi|}{|\eta|} \left(-\frac{\xi - \eta}{|\xi - \eta|} + \frac{\eta}{|\eta|} \frac{\eta^k}{|\eta|} \frac{(\xi - \eta)^k}{|\xi - \eta|} \right) \\ &= \frac{|\xi|}{|\eta|} \left(-\frac{\eta}{|\eta|} - \frac{\xi - \eta}{|\xi - \eta|} + \frac{\eta}{|\eta|} + \frac{\eta}{|\eta|} \frac{\eta^k}{|\eta|} \frac{(\xi - \eta)^k}{|\xi - \eta|} \right) \\ &= \frac{|\xi|}{|\eta|} \left(-\left(\frac{\eta}{|\eta|} + \frac{\xi - \eta}{|\xi - \eta|} \right) + \frac{\eta}{|\eta|} \frac{\eta^k}{|\eta|} \left(\frac{\eta^k}{|\eta|} + \frac{(\xi - \eta)^k}{|\xi - \eta|} \right) \right) \\ &= \frac{|\xi| - |\eta|}{|\eta|} \left(-\left(\frac{\eta}{|\eta|} + \frac{\xi - \eta}{|\xi - \eta|} \right) + \frac{\eta}{|\eta|} \frac{\eta^k}{|\eta|} \left(\frac{\eta^k}{|\eta|} + \frac{(\xi - \eta)^k}{|\xi - \eta|} \right) \right) \\ &\quad - \left(\frac{\eta}{|\eta|} + \frac{\xi - \eta}{|\xi - \eta|} \right) + \frac{\eta}{|\eta|} \frac{\eta^k}{|\eta|} \left(\frac{\eta^k}{|\eta|} + \frac{(\xi - \eta)^k}{|\xi - \eta|} \right) \\ &=: \frac{|\xi| - |\eta|}{|\eta|} \tilde{m}(\eta, \xi - \eta) + f(\eta, \xi - \eta). \end{aligned}$$

Now we define bilinear operators \tilde{T}_1, \tilde{T}_2 via

$$\tilde{T}_1(a, b)^\wedge(\xi) := \int f(\eta, \xi - \eta) a^\wedge(\eta) b^\wedge(\xi - \eta) d\eta,$$

$$\tilde{T}_2(a, b)^\wedge(\xi) := \int \frac{|\xi| - |\eta|}{|\eta|} \tilde{m}(\eta, \xi - \eta) a(\eta) b(\xi - \eta) d\eta.$$

By the commutator arguments of [CRW76], $\tilde{T}_1(\cdot, \cdot)$ can be estimated

$$\|\tilde{T}_1(h, b)\|_{\mathcal{H}} \lesssim \|h\|_2 \|b\|_2.$$

Indeed, for some constants c_1, c_2 ,

$$\tilde{T}_1(h, b) = c_1(\mathcal{R}[a] + \mathcal{R}[b]) + c_2(\mathcal{R}_k[\mathcal{R}\mathcal{R}_k[a]] b + \mathcal{R}\mathcal{R}_k[a] \mathcal{R}_k[b])$$

Thus,

$$\begin{aligned} \|\Pi_1 T(h, b)\|_{\mathcal{H}} &\lesssim \|\Pi_1(T(h, b) - \tilde{T}_1(h, b) - \tilde{T}_2(h, b))\|_{\mathcal{H}} + \|\Pi_1 \tilde{T}_2(h, b)\|_{\mathcal{H}} + \|h\|_2 \|b\|_2 \\ &=: I + II + \|h\|_2 \|b\|_2. \end{aligned}$$

The term I can be estimated in the same manner as in (4.29) for $\mu < 1$, since the only term for which (4.25) was not applicable is now cut away.

Thus it remains to estimate II . Note that \tilde{m} is a finite sum of products of zero-order multipliers of η and $\xi - \eta$, which thus plays no role in our argument. The crucial point is, that by (4.27)

$$\frac{|\xi| - |\eta|}{|\eta|} = \sum_{l=1}^{\infty} \frac{1}{l!} m_l(\eta) n_l(\xi - \eta) |\eta|^{-l} |\xi - \eta|^l,$$

so the exponent of $|\xi - \eta|$ is always strictly greater than zero. The estimate on II follows then from along the arguments in (4.29) for $\mu = 0$.

Estimate of Π_3

It remains to estimate Π_3 . Let N_1, N_2 be a CZ-zero-multiplier operator (in this case or the identity or the Riesz Transform \mathcal{R}). In order to estimate $\Pi_3 T(h, b)$ in the Hardy-space norm, it then suffices to control terms of the form

$$\|N_1 |\nabla|^\mu \Pi_3(N_2 h I_\mu b)\|_{\mathcal{H}}.$$

Again, for some ψ , $\|\psi\|_{\dot{B}_{\infty, \infty}^0} \leq 1$,

$$\begin{aligned} \|N_1 |\nabla|^\mu \Pi_3(N_2 h I_\mu b)\|_{\mathcal{H}} &\lesssim \| |\nabla|^\mu \Pi_3(N_2 h I_\mu b)\|_{\mathcal{H}} \\ &\lesssim \| |\nabla|^\mu \Pi_3(N_2 h I_\mu b)\|_{\dot{B}_{1,1}^0} \\ &\lesssim \int |\nabla|^\mu \Pi_3(N_2 h I_\mu b) \psi \\ &= \int \Pi_3(N_2 h I_\mu b) |\nabla|^\mu \psi \\ &= \sum_{j \in \mathbb{Z}} \sum_{l=j-4}^{j+4} \int N_2 h_j I_\mu b_l |\nabla|^\mu \psi \\ &= \sum_{j \in \mathbb{Z}} \sum_{l=j-4}^{j+4} \int N_2 h_j I_\mu b_l |\nabla|^\mu \psi^{j+6} \\ &\stackrel{(4.24)}{\lesssim} \sum_{j \in \mathbb{Z}} \sum_{l=j-4}^{j+4} \int |N_2 h_j| |I_\mu b_l| 2^{\mu j} \|\psi\|_{\dot{B}_{\infty, \infty}^0} \\ &\lesssim \|N_2 h\|_{\dot{F}_{2,2}^0} \|I_\mu b\|_{\dot{F}_{2,2}^\mu} \\ &\lesssim \|h\|_2 \|b\|_2. \end{aligned}$$

This shows (4.15) □

Then we are able to give the

Proof of (4.16). The estimate is a consequence of (4.15): Let $\psi \in C_0^\infty$, $\|\psi\|_2 \leq 1$ such that

$$\begin{aligned} \|\mathcal{C}(f, \mathcal{R})[|\nabla|^\mu \varphi]\|_2 &\leq 2 \int \mathcal{C}(f, \mathcal{R})[|\nabla|^\mu \varphi] \psi \\ &= 2 \int (f \mathcal{R}[|\nabla|^\mu \varphi] - \mathcal{R}[f |\nabla|^\mu \varphi]) \psi \\ &= -2 \int \varphi |\nabla|^\mu (\mathcal{R}[f \psi] - f \mathcal{R}[\psi]). \end{aligned}$$

Taking $h := \psi$ and $b := |\nabla|^\mu f$, one concludes the proof of (4.16) by (4.15). □

Proof of (4.17). If $\mu = 1$, (4.17) follows from the following easy argument:

$$\|H_1(\varphi, g)\|_2 = c \|\mathcal{R}_i \mathcal{R}_i H_1(\varphi, g)\|_2 \lesssim \sum_i \|\mathcal{R}_i H_1(\varphi, g)\|_2.$$

Moreover,

$$\begin{aligned} \mathcal{R}_i H_1(\varphi, g) &= g \partial_i \varphi + \varphi \partial_i g - \mathcal{R}_i[g |\nabla|^1 \varphi] - \mathcal{R}_i[\varphi |\nabla|^1 g] \\ &= g \mathcal{R}_i[|\nabla|^1 \varphi] + \varphi \mathcal{R}_i[|\nabla|^1 g] - \mathcal{R}_i[g |\nabla|^1 \varphi] - \mathcal{R}_i[\varphi |\nabla|^1 g] \\ &= \mathcal{C}(g, \mathcal{R}_i)[|\nabla|^1 \varphi] + \mathcal{C}(\varphi, \mathcal{R}_i)[|\nabla|^1 g]. \end{aligned}$$

Consequently, by the [CRW76]-commutator theorem

$$\|H_1(\varphi, g)\|_2 \lesssim \sum_i \|\mathcal{C}(g, \mathcal{R}_i)[|\nabla|^1 \varphi]\|_2 + \sum_i \|\mathcal{C}(\varphi, \mathcal{R}_i)[|\nabla|^1 g]\|_2 \lesssim \sum_i \|\mathcal{C}(g, \mathcal{R}_i)[|\nabla|^1 \varphi]\|_2 + [\varphi]_{BMO} \| |\nabla|^1 g \|_2,$$

and therefore we have reduced (4.17) to (4.16).

If $\mu < 1$, we show that for

$$\tilde{H}_\mu(a, b) := |\nabla|^\mu a b - a |\nabla|^\mu b - |\nabla|^\mu(ab)$$

we have the estimate

$$\|\tilde{H}_\mu(a, b)\|_{\mathcal{H}} \lesssim \|a\|_2 \| |\nabla|^\mu b \|_2.$$

Use again the decomposition in Π_1, Π_2, Π_3 . We have for any $\mu > 0$,

$$\|\Pi_3(|\nabla|^\mu(ab))\|_{\mathcal{H}} + \|\Pi_1(a|\nabla|^\mu b)\|_{\mathcal{H}} + \|\Pi_2((|\nabla|^\mu a) b)\|_{\mathcal{H}} + \|\Pi_2(a|\nabla|^\mu b)\|_{\mathcal{H}} \leq \|a\|_2 \| |\nabla|^\mu b \|_2, \quad (4.30)$$

Estimate of Π_1 . We have,

$$\|\Pi_1 \tilde{H}_\mu(a, b)\|_{\mathcal{H}} \stackrel{(4.30)}{\lesssim} \left\| \sum_j ((|\nabla|^\mu a_j) b^{j-4} - |\nabla|^\mu(a_j b^{j-4})) \right\|_{\mathcal{H}} + \|a\|_2 \| |\nabla|^\mu b \|_2.$$

Thus we have to estimate

$$\sum_{j=-\infty}^{\infty} \int_{\mathbb{R}^m} ((|\nabla|^\mu a_j) b^{j-4} - |\nabla|^\mu(a_j b^{j-4})) \psi_{\tilde{j}}$$

The respective kernel is

$$k(\xi, \eta) = |\eta|^\mu - |\xi|^\mu = \sum_{l=1}^{\infty} c_l m_l(\eta) n_l(\xi - \eta) |\eta|^{\mu-l} |\xi - \eta|^l,$$

that is

$$\begin{aligned} & \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}^m} ((|\nabla|^\mu a_j) b^{j-4} - |\nabla|^\mu(a_j b^{j-4})) \psi_{\tilde{j}} \\ \stackrel{\mu < 1}{=} & \sum_{l=1}^{\infty} c_l 2^{4(\mu-l)} \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}^m} 2^{j(l-\mu)} (N_l I_{l-\mu} a_j) 2^{(\mu-l)(j-4)} |\nabla|^{l-\mu} |\nabla|^\mu b^{j-4} \psi_{\tilde{j}} \\ \stackrel{\mu < 1}{\lesssim} & \|a\|_2 \| |\nabla|^\mu b \|_2, \end{aligned}$$

where for the last step we can just copy the remaining arguments from (4.29), since $\mu < 1 \leq l$.

Estimate of Π_2 . Since

$$\tilde{H}_\mu(a, b) = |\nabla|^\mu a b - 2a |\nabla|^\mu b + (a |\nabla|^\mu b - |\nabla|^\mu(ab)),$$

we can

$$\|\Pi_2 \tilde{H}_\mu(a, b)\|_{\mathcal{H}} \stackrel{(4.30)}{\lesssim} \left\| \sum_j (|\nabla|^\mu(a^{j-4} b_j) - a^{j-4} |\nabla|^\mu b_j) \right\|_{\mathcal{H}} + \|a\|_2 \| |\nabla|^\mu b \|_2.$$

Since $\mu < 1$, as in the argument for Π_1 , we can estimate

$$\left\| \sum_j (|\nabla|^\mu(a^{j-4} b_j) - a^{j-4} |\nabla|^\mu b_j) \right\|_{\mathcal{H}} \lesssim \|a\|_2 \| |\nabla|^\mu b \|_2.$$

Estimate of Π_3 . Finally, we have

$$\|\Pi_3 \tilde{H}_\mu(a, b)\|_{\mathcal{H}} \stackrel{(4.30)}{\lesssim} \left\| \sum_{j \approx i} (|\nabla|^\mu(a_j) b_i - a_j |\nabla|^\mu b_i) \right\|_{\mathcal{H}} + \|a\|_2 \|\nabla|^\mu b\|_2,$$

and have to estimate

$$\sum_{j=-\infty}^{\infty} \sum_{i=j-4}^{j+4} (|\nabla|^\mu(a_j) b_i - a_j |\nabla|^\mu b_i) \psi^{j-6} + \sum_{j=-\infty}^{\infty} \sum_{i=j-4}^{j+4} \sum_{k=j-6}^{j+6} (|\nabla|^\mu(a_j) b_i - a_j |\nabla|^\mu b_i) \psi_k =: I + II,$$

The kernel here is (note that for $\xi \in \text{supp}(\psi^{j-6})^\wedge$, $\eta \in \text{supp}(a_j)^\wedge$, $|\xi|/|\eta| \leq \frac{1}{2}$ which ensures convergence)

$$k(\eta, \xi) = |\eta|^\mu - |\xi - \eta|^\mu = \sum_{l=1}^{\infty} c_l m_l(\xi - \eta) n_l(\xi) |\eta|^{\mu-l} |\xi|^l, \quad (4.31)$$

thus we have

$$\begin{aligned} I &= \sum_{l=1}^{\infty} c_l \sum_{j=-\infty}^{\infty} \sum_{i=j-4}^{j+4} \int M_l |\nabla|^{\mu-l} a_j b_i N_l |\nabla|^l \psi^{j-6} \\ &\lesssim \sum_{l=1}^{\infty} c_l 2^{-6l} \sup_k 2^{-l(k-6)} \|N_l |\nabla|^l \psi^{k-6}\|_{\infty} \sum_{j=-\infty}^{\infty} \sum_{i=j-4}^{j+4} \int \left| 2^{(l-\mu)j} N_l |\nabla|^{\mu-l} a_j \right| |2^{\mu i} b_i| \\ &\stackrel{(4.24)}{\lesssim} \|\psi\|_{\dot{B}_{\infty, \infty}^0} \sum_{l=1}^{\infty} c_l 2^{-6l} \sum_{j=-\infty}^{\infty} \sum_{i=j-4}^{j+4} \int \left| 2^{(l-\mu)j} M_l |\nabla|^{\mu-l} a_j \right| |2^{\mu i} b_i| \\ &\lesssim \|\psi\|_{\dot{B}_{\infty, \infty}^0} \|a\|_2 \|\nabla|^\mu b\|_2. \end{aligned}$$

For II , where the expansion (4.31) might not be convergent, the situation is even easier,

$$\begin{aligned} |II| &\lesssim \sum_{j=-\infty}^{\infty} \sum_{i=j-4}^{j+4} \sum_{k=j-6}^{j+6} \int (|2^{-\mu j} |\nabla|^\mu a_j| |2^{\mu j} b_i| + |a_j| \|\nabla|^\mu b_i\|) |\psi_k| \\ &\lesssim \|\psi\|_{\dot{B}_{\infty, \infty}^0} \left(\left(\int \sum_j |2^{-\mu j} |\nabla|^\mu a_j|^2 \right)^{\frac{1}{2}} \left(\int \sum_j |2^{\mu j} b_i|^2 \right)^{\frac{1}{2}} + \left(\int |a_j|^2 \right)^{\frac{1}{2}} \left(\int \|\nabla|^\mu b_i\|^2 \right)^{\frac{1}{2}} \right) \\ &\lesssim \|\psi\|_{\dot{B}_{\infty, \infty}^0} \|a\|_2 \|\nabla|^\mu b\|_2. \end{aligned}$$

□

Proof of (4.18) for $\mu = 1$. We have (by the classical product rule, or equivalently expanding the symbols $|\xi|^2 = |\xi - \eta|^2 + |\eta|^2 + 2\xi \cdot (\xi - \eta)$). With \mathcal{R}_k we will by a abuse of notation call the linear operators with symbol $\xi^k/|\xi|$. By the classical product rule for $\mathcal{R}_i |\nabla|^1 = c \partial_i$

$$\begin{aligned} \mathcal{R}_i H_1(a, b) &= a \mathcal{R}_i |\nabla|^1 b + b \mathcal{R}_i |\nabla|^1 a - \mathcal{R}_i(a |\nabla|^1 b) - \mathcal{R}_i(b |\nabla|^1 a) \\ &= I_1(|\nabla|^1 a) \mathcal{R}_i |\nabla|^1 b + I_1(|\nabla|^1 b) \mathcal{R}_i |\nabla|^1 a - \mathcal{R}_i(I_1(|\nabla|^1 a) |\nabla|^1 b) - \mathcal{R}_i(I_1(|\nabla|^1 b) |\nabla|^1 a) \\ &= I_1(|\nabla|^1 a) \mathcal{R}_i |\nabla|^1 b - \mathcal{R}_i(I_1(|\nabla|^1 a) |\nabla|^1 b) \\ &\quad + I_1(|\nabla|^1 b) \mathcal{R}_i |\nabla|^1 a - \mathcal{R}_i(I_1(|\nabla|^1 b) |\nabla|^1 a) \end{aligned}$$

Thus, $\mathcal{R}_i |\nabla|^1 H_1(a, b)$ can be estimated via (4.15), and as this holds for any $i \in \mathbb{N}$, we have (4.18) for $\mu = 1$. □

5. Energy approach for optimal frame: Proof of Theorem 1.6

In this section we construct a suitable frame P for our equation, transforming the *antisymmetric* (essentially) L^2 -potential $\Omega[\cdot]$ into an $L^{2,1}$ - or even better in an $I_\mu \mathcal{H}$ -potential $\Omega^P[\cdot]$. Here, \mathcal{H} is the Hardy space, and with the previous statement we essentially mean that

$$\int \Omega^P[f] \leq C_{\Omega^P} \|f\|_{(2,\infty)}, \quad \text{or} \quad \int \Omega^P[|\nabla|^\mu \varphi] \leq C_{\Omega^P} \|\varphi\|_{BMO}, \quad \text{respectively,} \quad (5.1)$$

where BMO is the space dual to \mathcal{H} . This is an improvement, since for the non-transformed Ω , we only had the estimate

$$\int \Omega[f] \leq C_\Omega \|f\|_2. \quad (5.2)$$

For motivation of the arguments presented here, let us recall the classical setting [Riv07], where we have the equation (usually for $w^i := \nabla u^i \in L^2(\mathbb{R}^m, \mathbb{R}^2)$)

$$-\operatorname{div}(w^i) = \tilde{\Omega}_{ik} \cdot w^l,$$

for $\tilde{\Omega}_{ik} = -\tilde{\Omega}_{ki} \in L^2(\mathbb{R}^m, \mathbb{R}^2)$, and we look for an orthogonal transformation $P \in W^{1,2}(\mathbb{R}^m, SO(N))$, $SO(N) \subset \mathbb{R}^{N \times N}$ being the special orthogonal group, such that

$$\int \tilde{\Omega}_{ik}^P \cdot \nabla \varphi = 0, \quad (5.3)$$

where

$$\tilde{\Omega}_{ij}^P = P_{ik} \nabla P_{kj}^T + P_{ik} \tilde{\Omega}_{kl} P_{lj}^T, \quad \text{or equivalently,} \quad -\operatorname{div}(P_{il} w^l) = \tilde{\Omega}_{ik}^P \cdot P_{kl} w^l.$$

Also in this case, the estimate (5.3) is an improvement from the estimate for the non-transformed $\tilde{\Omega}$

$$\int \tilde{\Omega} \cdot \nabla \varphi \leq C_{\tilde{\Omega}} \|\nabla \varphi\|_2,$$

philosophically similar to the improvement (5.1) from the starting point (5.2).

For the construction of P such that (5.3) holds, there are two different arguments known: Rivière [Riv07] adapted a result by Uhlenbeck [Uhl82] which is based on the continuity method (for the set $t\Omega$, $t \in [0, 1]$) and relies on non-elementary a-priori estimates for $\tilde{\Omega}^P$ which also needs L^2 -smallness of $\tilde{\Omega}$. In [Sch10a] we proposed to use arguments from Hélein's moving frame method [Hél91]: Then the construction of P relies on the fact that (5.3) is the Euler-Lagrange equation of the energy

$$\tilde{E}(Q) := \|\tilde{\Omega}^Q\|_2^2, \quad Q \in SO(N), \text{ a.e.}, \quad (5.4)$$

the minimizer of which exists by the elementary direct method.

Both construction arguments have been generalized to the fractional setting for $\Omega[\cdot] \equiv \Omega \cdot$ a pointwise multiplication-operator [DLR11a, Sch11]. In our situation, where $\Omega[\cdot]$ is allowed to be a linear bounded operator from L^2 to L^1 , we adapt the argument in [Hél91, Sch10a, Sch11], and minimize essentially the energy

$$E(Q) := \sup_{\psi \in L^2} \int \Omega^Q[\psi], \quad Q \in SO(N), \text{ a.e.}$$

While the construction of a minimizer of E , see Lemma 5.5, is not much more difficult as in the earlier situations [Hél91, Sch10a, Sch11], when computing the Euler-Lagrange equations, see Lemma 5.6, we have several error terms, which stem from commutators of the form $f\Omega[g] - \Omega[fg]$, which are trivial if $\Omega[\cdot]$ is a pointwise-multiplication operator $\Omega[\cdot] = \Omega \cdot$. In Lemma 5.7 we then show that these error terms all behave well enough, if we take the for us relevant case of $\Omega[\cdot]$ being of the form $\mathcal{AR}[\cdot]$.

5.1. Preliminary propositions

Here we recall some elementary statements, which enter the proof of Theorem 1.6. Proposition 5.1 and Proposition 5.2 are simple duality arguments for linear, bounded mappings between Banach spaces. Proposition 5.4 is a quantified embedding from L^1 into BMO .

Proposition 5.1. For any $s > 0$ there exists $\Lambda_0, C_s > 1$ such that the following holds: Let $f \in L^2(\mathbb{R}^m)$, $|\nabla|^s f \in L^2(\mathbb{R}^m)$ and assume $f \equiv 0$ on $\mathbb{R}^m \setminus B_r$ for some $B_r \subset \mathbb{R}^m$. Then for any $\Lambda \geq \Lambda_0$,

$$\| |\nabla|^s f \|_{2, \mathbb{R}^m \setminus B_{\Lambda r}} \leq C_s \Lambda^{-\frac{m}{2}-s} \| |\nabla|^s f \|_{2, B_{\Lambda r}}.$$

Proof. Using Corollary B.2,

$$\| |\nabla|^s f \|_{2, \mathbb{R}^m \setminus B_{\Lambda r}} \lesssim \Lambda^{-\frac{m}{2}-s} \| |\nabla|^s f \|_{2, \mathbb{R}^m \setminus B_{\Lambda r}} + \Lambda^{-\frac{m}{2}-s} \| |\nabla|^s f \|_{2, B_{\Lambda r}}.$$

Thus, if $\Lambda > \Lambda_0$ for a Λ_0 depending only on s , we can absorb and conclude. \square

Proposition 5.2. Let $A : L^2(\mathbb{R}^m) \rightarrow L^1(\mathbb{R}^m)$ be a linear, bounded operator. Then there exists $\bar{g} \in L^2(\mathbb{R}^m)$, $\|\bar{g}\|_{2, \mathbb{R}^m} = 1$ such that

$$\sup_{\|\psi\|_{2, \mathbb{R}^m} \leq 1} \int A[\psi] = \int A[\bar{g}].$$

In particular (taking instead of A the operator $\tilde{A} := A[\chi_D \cdot]$, for any $D \subset \mathbb{R}^m$ there exists $\bar{g}_D \in L^2(D)$, $\|\bar{g}_D\|_{2, D} \leq 1$, $\text{supp } \bar{g} \subset \bar{D}$, such that

$$\sup_{\|\psi\|_{2, \mathbb{R}^m} \leq 1, \text{supp } \psi \subset \bar{D}} \int A[\psi] = \int A[\bar{g}_D].$$

Proof. As $f^* \in (L^2(\mathbb{R}^m))^*$ for

$$f^*(\psi) := \int A[\psi]$$

is a linear bounded functional, there exists a unique $\bar{f} \in L^2(\mathbb{R}^m)$ such that

$$\int A[\psi] = \int \bar{f} \psi, \tag{5.5}$$

and

$$\|\bar{f}\|_{2, \mathbb{R}^m} = \sup_{\|\psi\|_{2, \mathbb{R}^m} \leq 1} \int A[\psi].$$

On the other hand,

$$\|\bar{f}\|_{2, \mathbb{R}^m}^2 = \int \bar{f} \bar{f} \stackrel{(5.5)}{=} \int A[\bar{f}],$$

which in turn implies that for $\bar{g} := \|\bar{f}\|_2^{-1} \bar{f}$,

$$\sup_{\|\psi\|_{2, \mathbb{R}^m} \leq 1} \int A[\psi] = \|\bar{f}\|_{2, \mathbb{R}^m} = \int A[\bar{g}].$$

\square

Proposition 5.3. Let $A : L^2(\mathbb{R}^m) \rightarrow L^1(\mathbb{R}^m)$ be a linear, bounded operator. Then there exists a linear, bounded operator $A^* : L^\infty(\mathbb{R}^m) \rightarrow L^2(\mathbb{R}^m)$ such that

$$\int g A[f] = \int f A^*[g] \quad \text{for any } f \in L^2(\mathbb{R}^m), g \in L^\infty(\mathbb{R}^m).$$

Moreover, $\bar{g} = \|A(1)\|_2^{-1} A^*(1)$ for the \bar{g} from Proposition 5.2.

Proof. For any $g \in L^\infty(\mathbb{R}^m)$, we have

$$T_g[\cdot] := \int g A[\cdot] \in (L^2(\mathbb{R}^m))^*,$$

thus there exists a unique $A_g^* \in L^2(\mathbb{R}^m)$ such that

$$T_g[f] = \int A_g^* f \quad \text{for all } f \in L^2(\mathbb{R}^m).$$

For $\lambda_1, \lambda_2 \in \mathbb{R}$ and $g_1, g_2 \in L^\infty$, and any $f \in L^2(\mathbb{R}^m)$

$$\int A_{\lambda_1 g_1 + \lambda_2 g_2}^* f = \lambda_1 \int g_1 A[f] + \lambda_2 \int g_2 A[f] = \lambda_1 \int A_{g_1} f + \lambda_2 \int A_{g_2} f,$$

which implies that $A^*[\cdot] := A^*$ is linear. As for boundedness, note that

$$\|A^*[g]\|_{2, \mathbb{R}^m} = \sup_{\|f\|_{2, \mathbb{R}^m} \leq 1} \int A^*[g]f = \sup_{\|f\|_{2, \mathbb{R}^m} \leq 1} \int gA[f] \leq \|g\|_\infty \|A\|_{L^2 \rightarrow L^1},$$

that is

$$\|A^*\|_{L^\infty \rightarrow L^2} \leq \|A\|_{L^2 \rightarrow L^1}.$$

Finally, we have that

$$\int \bar{f} \psi \stackrel{(5.5)}{=} \int A[\psi] = \int A^*[1] \psi,$$

which - given that it holds for any $\psi \in L^2(\mathbb{R}^m)$ - implies $\bar{f} = A^*[1]$. \square

Proposition 5.4. *Let $\varphi \in C_0^\infty(B_r)$, then*

$$\|\varphi\|_1 \leq C_m r^m [\varphi]_{BMO}.$$

Proof. Let $\Lambda > 0$, then

$$\|\varphi\|_{1, B_r} \leq \|\varphi - |B_{\Lambda r}|^{-1} \int \varphi\|_{B_r} + \frac{|B_r|}{|B_{\Lambda r}|} \|\varphi\|_{1, B_r},$$

and consequently for, say, $\Lambda = 2$,

$$\|\varphi\|_{1, B_r} \leq 2\|\varphi - |B_{\Lambda r}|^{-1} \int \varphi\|_{B_r} \lesssim (\Lambda r)^m |B_{\Lambda r}|^{-1} \|\varphi - |B_{\Lambda r}|^{-1} \int \varphi\|_{B_{\Lambda r}} \lesssim (\Lambda r)^m [\varphi]_{BMO}.$$

\square

5.2. Energy with potentials

Let $\Omega^{i,j} : L^2(\mathbb{R}^m) \rightarrow L^1(\mathbb{R}^m)$, $1 \leq i, j \leq N$ be a linear bounded Operator. And set

$$\Omega_{ij}^Q[f] := (|\nabla|^\mu (Q - I)_{ik}) Q_{kj}^T f + Q_{ik} \Omega_{kl} [Q_{lj}^T f],$$

for $\text{supp}(Q - I) \subset B_r$, $|\nabla|^\mu Q \in L^2(\mathbb{R}^{N \times N})$ and $Q \in SO(N)$ almost everywhere. For $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^{N \times N}$, we write

$$\Omega^Q[\psi] := (|\nabla|^\mu (Q - I)_{ik}) Q_{kj}^T \psi_{ij} + Q_{ik} \Omega_{kl} [Q_{lj}^T \psi_{ij}],$$

Having in mind (5.4), we then define the energy

$$E(Q) \equiv E_{r,x,\Lambda,s,2}(Q) := \begin{cases} \sup_{\substack{\psi \in C_0^\infty(B_{\Lambda r}(x), \mathbb{R}^{N \times N}) \\ \|\psi\|_2 \leq 1}} \int_{\mathbb{R}^m} (\Omega^Q)[\psi] & \text{if } \text{supp}(Q - I) \subset \overline{B_r(x)}, \\ \infty & \text{else.} \end{cases} \quad (5.6)$$

Obviously, $Q \equiv I$ is admissible and $E(I) < \infty$. Thus there exists a minimizing sequence, and one can hope for a minimizer:

Lemma 5.5. *For any $\mu > 0$ there exists $\Lambda_0 > 1$ such that for any $\Lambda \geq \Lambda_0$, the following holds: There exists an admissible function P for E such that $E(P) \leq E(Q)$ for any other admissible function Q . Moreover,*

$$\|\nabla|^\mu P\|_{2, B_{\Lambda r}(x)} + \Lambda^{\frac{m}{2} + \mu} \|\nabla|^\mu P\|_{2, \mathbb{R}^m \setminus B_{\Lambda r}(x)} \leq C_\mu \|\Omega\|_{2 \rightarrow 1, B_{\Lambda r}(x)}. \quad (5.7)$$

Here,

$$\|\Omega\|_{2 \rightarrow 1, D} := \sup_{\psi \in C_0^\infty(D, \mathbb{R}^{N \times N}), \|\psi\|_2 \leq 1} \|\Omega[\psi]\|_1$$

Proof. Take Λ_0 from Proposition 5.1 and assume $\Lambda \geq \Lambda_0$. We have for any $\psi \in C_0^\infty(B_{\Lambda r}, \mathbb{R}^{N \times N})$, $\|\psi\|_2 \leq 1$

$$\begin{aligned} E(Q) &\geq \int (|\nabla|^\mu(Q - I) Q^T)_{ij} \psi_{ij} + \int Q \Omega [Q^T \psi] \\ &\geq \int (|\nabla|^\mu(Q - I) Q^T)_{ij} \psi_{ij} - \|\Omega\|_{2 \rightarrow 1, B_{\Lambda r}}, \end{aligned}$$

which (taking the supremum over such ψ) implies

$$\|\nabla|^\mu(Q - I)\|_{2, B_{\Lambda r}} \leq E(Q) + \|\Omega\|_{2 \rightarrow 1, B_{\Lambda r}}.$$

According to Proposition 5.1, this implies (as $Q \equiv I$ on $\mathbb{R}^n \setminus B_r$),

$$\|\nabla|^\mu(Q - I)\|_{2, \mathbb{R}^m} \leq C_\mu (E(Q) + \|\Omega\|_{2 \rightarrow 1, B_{\Lambda r}}).$$

Consequently, for a minimizing sequence P_k ,

$$\|\nabla|^\mu(P_k - I)\|_{2, \mathbb{R}^m} \leq C_\mu \|\Omega\|_{2 \rightarrow 1, B_{\Lambda r}},$$

and up to taking a subsequence, we may assume that there is an admissible function P such that $|\nabla|^\mu P_k$ converges L^2 -weakly to $|\nabla|^\mu P$ and P_k converges pointwise and strongly to P .

Then, for any fixed $\psi \in C_0^\infty(B_{\Lambda r})$, $\|\psi\|_{2, \mathbb{R}^{N \times N}} \leq 1$

$$E(P_k) \geq \int \Omega^P[\psi] + \int \Omega^{P_k}[\psi] - \Omega^P[\psi].$$

We claim that

$$\int \Omega^{P_k}[\psi] - \Omega^P[\psi] \xrightarrow{k \rightarrow \infty} 0, \quad (5.8)$$

which, once proven, implies that

$$\inf E(\cdot) \geq \int \Omega^P \psi,$$

which by the arbitrary choice of ψ implies that P is a minimizer. In order to show (5.8), note that

$$\begin{aligned} \Omega^{P_k}[\psi] - \Omega^P[\psi] &= |\nabla|^\mu P_k (P_k^T - P^T) \psi + |\nabla|^\mu (P_k - P) P^T \psi + (P_k - P) \Omega [P_k^T \psi] + P \Omega [(P_k^T - P^T) \psi] \\ &=: I_k + II_k + III_k + IV_k. \end{aligned}$$

Since $|P_k|, |P| \leq 1$, all terms of the form $(P_k^T - P^T) \psi \xrightarrow{k \rightarrow \infty} 0$ in L^2 , by Lebesgue's dominated convergence. Thus, $\int I_k + \int IV_k \xrightarrow{k \rightarrow \infty} 0$. By the weak L^2 -convergence of $|\nabla|^\mu P_k$, also $\int II_k \xrightarrow{k \rightarrow \infty} 0$. Since $P_k^T \psi \rightarrow P^T \psi$ in $L^2(\mathbb{R}^m)$, also $\int III_k \xrightarrow{k \rightarrow \infty} \int \Omega [P^T \psi]$ in L^1 and in particular pointwise almost everywhere. Then also $\int III_k \xrightarrow{k \rightarrow \infty} 0$. \square

Lemma 5.6. *Let P be a minimizer of $E(\cdot)$ as in (5.6), and assume that*

$$\Omega_{ij}[\cdot] = -\Omega_{ji}[\cdot] \quad 1 \leq i, j \leq N. \quad (5.9)$$

Then for any $\varphi \in C_0^\infty(B_r(x))$,

$$\begin{aligned} - \int \Omega^P [|\nabla|^\mu \varphi] &= \frac{1}{2} \int H_\mu(P - I, P^T - I) |\nabla|^\mu \varphi \\ &\quad - \int so(PC(\varphi, \Omega) [P^T \overline{\Omega}^{P^T} \chi_{D_\Lambda}]) \\ &\quad + \int so(\overline{\Omega}^P \chi_{D_\Lambda} P H_\mu(\varphi, P^T - I)) \\ &\quad - \int so(C(P, \Omega) [|\nabla|^\mu \varphi] P^T) \\ &\quad + \int \Omega^P [(1 - \chi_{D_\Lambda}) |\nabla|^\mu \varphi]. \end{aligned}$$

Here, we denote for a matrix $A \in \mathbb{R}^{N \times N}$, the antisymmetric part with $so(A) = 2^{-1}(A - A^T)$, and for a mapping $g: L^2 \rightarrow L^1$, we denote \bar{g} as in Proposition 5.2.

Proof. We set $D := B_r(x)$ and $D_\Lambda := B_{\Lambda r}(x)$. Let $\varphi \in C_0^\infty(D)$, $\omega \in so(N)$. We distort the minimizer P of $E(\cdot)$ by

$$Q_\varepsilon := e^{\varepsilon\varphi\omega} P = P + \varepsilon\varphi\omega P + o(\varepsilon) \in H_I^{\frac{n}{2}}(D, SO(N)),$$

that is we know that

$$E(Q_\varepsilon) - E(P) \geq 0 \quad (5.10)$$

We compute

$$\begin{aligned} & |\nabla|^\mu(Q_\varepsilon - I) Q^T \\ &= |\nabla|^\mu(P - I) P^T + \varepsilon\varphi(\omega |\nabla|^\mu(P - I) P^T - |\nabla|^\mu(P - I) P^T \omega) + \varepsilon|\nabla|^\mu\varphi\omega + \varepsilon\omega H_\mu(\varphi, P - I)P^T + o(\varepsilon), \end{aligned} \quad (5.11)$$

and

$$Q_\varepsilon\Omega [Q_\varepsilon^T \cdot] = P\Omega [P^T \cdot] + \varepsilon(\varphi\omega P\Omega [P^T \cdot] - P\Omega [P^T \omega\varphi \cdot]) + o(\varepsilon). \quad (5.12)$$

Together, we infer from (5.11) and (5.12) (denoting the Hilbert-Schmidt matrix product $A : B := A_{ij}B_{ij}$)

$$\Omega^{Q_\varepsilon}[\psi] = \Omega^P[\psi] + \varepsilon(\varphi\omega \Omega^P[\psi] - \Omega^P[\omega\psi\varphi]) + \varepsilon|\nabla|^\mu\varphi\omega : \psi + \varepsilon\omega H_\mu(\varphi, P - I)P^T : \psi + o(\varepsilon)[\psi].$$

Thus, for any $\varepsilon > 0$, $\psi \in C_0^\infty(D_\Lambda, \mathbb{R}^{N \times N})$, $\|\psi\|_2 \leq 1$,

$$\begin{aligned} \frac{1}{\varepsilon}(E(Q_\varepsilon) - E(P)) &\geq \frac{1}{\varepsilon} \left(\int \Omega^P[\psi] - E(P) \right) \\ &\quad + \int (\varphi\omega \Omega^P[\psi] - \Omega^P[\omega\psi\varphi]) \\ &\quad + \int |\nabla|^\mu\varphi\omega : \psi \\ &\quad + \int \omega H_\mu(\varphi, P - I)P^T : \psi \\ &\quad + o(1). \end{aligned}$$

Let $\bar{\psi} \in L^2(D_\Lambda)$ such that $E(P) = \int \Omega^P[\bar{\psi}]$ (cf. Proposition 5.2), this implies for the choice $\psi := \bar{\psi}$

$$\begin{aligned} 0 &\stackrel{(5.10)}{\geq} \frac{1}{\varepsilon}(E(Q_\varepsilon) - E(P)) \geq \int (\varphi\omega \Omega^P[\bar{\psi}] - \Omega^P[\omega\bar{\psi}\varphi]) \\ &\quad + \int |\nabla|^\mu\varphi\omega : \bar{\psi} \\ &\quad + \int \omega H_\mu(\varphi, P - I)P^T : \bar{\psi} \\ &\quad + o(1). \end{aligned}$$

Letting $\varepsilon \rightarrow 0$, we then have

$$\begin{aligned} - \int |\nabla|^\mu\varphi\omega : \bar{\psi} &\geq \int \varphi\omega \Omega^P[\bar{\psi}] - \Omega^P[\omega\bar{\psi}\varphi] \\ &\quad + \int \omega H_\mu(\varphi, P - I)P^T : \bar{\psi} \end{aligned}$$

which holds for any $\varphi \in C_0^\infty(B_r)$. Replacing φ by $-\varphi$, we arrive at

$$- \int |\nabla|^\mu\varphi\omega : \bar{\psi} = \int \varphi\omega \Omega^P[\bar{\psi}] - \Omega^P[\omega\bar{\psi}\varphi] + \int \omega H_\mu(\varphi, P - I)P^T : \bar{\psi}. \quad (5.13)$$

Now we need to be more specific about the characteristics of $\bar{\psi}$. We have

$$E(P) = \sup_\psi \int \Omega^P[\psi] = \sup_\psi \int |\nabla|^\mu P_{ik} P_{kj}^T \psi_{ij} + P_{ik}\Omega_{kl}[P_{lj}^T \psi_{ij}].$$

Let $\Omega_{kl}^* : L^\infty(\mathbb{R}^m) \rightarrow L^2(\mathbb{R}^m)$ be the linear bounded operator such that (cf. Proposition 5.3)

$$\int_{\mathbb{R}^m} g \Omega_{kl}[f] = \int_{\mathbb{R}^m} \Omega_{kl}^*[g] f, \quad \text{for any } f \in L^2(\mathbb{R}^m), g \in L^\infty(\mathbb{R}^m).$$

Set then,

$$\left((\Omega^P)^* \right)_{ij}[f] := |\nabla|^\mu P_{ik} P_{kj}^T f + \Omega_{kl}^*[f P_{ik}] P_{lj}^T \in L^2(\mathbb{R}^m),$$

and

$$\left(\overline{\Omega^P} \right)_{ij} := \left((\Omega^P)^* \right)_{ij}[1] \in L^2(\mathbb{R}^m).$$

Since

$$\int g \left(\Omega^P \right)_{ij}[f] = \int \left((\Omega^P)^* \right)_{ij}[g] f \quad \text{for all } f \in L^2(\mathbb{R}^m), g \in L^\infty(\mathbb{R}^m),$$

we have

$$E(P) = \sup_{\psi} \int \overline{\Omega^P} : \psi \chi_{D_\Lambda} = c \int \overline{\Omega^P} : \overline{\Omega^P} \chi_{D_\Lambda} = c \int \Omega^P[\overline{\Omega^P} \chi_{D_\Lambda}],$$

for some normalizing constant c . That is,

$$(E(P))^2 = \int \left(\Omega^P \right)_{ij}[\chi_{D_\Lambda} \overline{\Omega^P}_{ij}],$$

and we can assume $\overline{\psi} = c \chi_{D_\Lambda} \overline{\Omega^P} = c \chi_{D_\Lambda} \overline{\Omega^P}$ for some normalizing constant c . Now,

$$- \int |\nabla|^\mu \varphi \omega : \overline{\psi} = -c \int |\nabla|^\mu \varphi \omega_{ij} \chi_{D_\Lambda} \overline{\Omega^P}_{ij} = -\omega_{ij} \int \Omega_{ij}^P[|\nabla|^\mu \varphi] + \int \omega : \Omega^P[(1 - \chi_{D_\Lambda})|\nabla|^\mu \varphi].$$

Consequently, (5.13) reads as

$$\begin{aligned} - \int \omega : \Omega^P[|\nabla|^\mu \varphi] &= \int \varphi \omega_{ik} \left(\Omega^P \right)_{kj}[\left(\overline{\Omega^P} \right)_{ij} \chi_{D_\Lambda}] - \left(\Omega^P \right)_{ij}[\omega_{ik} \left(\overline{\Omega^P} \right)_{kj} \varphi] \\ &\quad + \int_{D_\Lambda} \omega H_\mu(\varphi, P - I) P^T : \overline{\Omega^P} \\ &\quad + \int \omega : \Omega^P[(1 - \chi_{D_\Lambda})|\nabla|^\mu \varphi]. \end{aligned}$$

Note that, since $\varphi \in C_0^\infty(\mathbb{R}^m) \subset L^\infty$,

$$\omega_{ik} \int \left(\Omega^P \right)_{ij}[\left(\overline{\Omega^P} \right)_{kj} \varphi] = \omega_{ik} \int \left(\overline{\Omega^P} \right)_{ij} \left(\overline{\Omega^P} \right)_{kj} \varphi \stackrel{\omega \in \text{so}}{=} 0. \quad (5.14)$$

By the same argument,

$$\omega_{ik} \int \varphi \left(\Omega^P \right)_{kj}[\chi_{D_\Lambda} \left(\overline{\Omega^P} \right)_{ij}] = \omega_{ik} \int \varphi \left(\Omega^P \right)_{kj}[\chi_{D_\Lambda} \left(\overline{\Omega^P} \right)_{ij}] - \omega_{ik} \int \left(\Omega^P \right)_{kj}[\left(\overline{\Omega^P} \right)_{ij} \varphi]$$

and

$$\begin{aligned} &\omega_{ik} \int \varphi \left(\Omega^P \right)_{kj}[\left(\overline{\Omega^P} \right)_{ij} \chi_{D_\Lambda}] \\ &= \omega_{ik} \int \left((\Omega^P)^* \right)_{kj}[\varphi] \left(\overline{\Omega^P} \right)_{ij} \chi_{D_\Lambda} \\ &= \omega_{ik} \int \varphi \left((\Omega^P)^* \right)_{kj}[1] \left(\overline{\Omega^P} \right)_{ij} \chi_{D_\Lambda} - \omega_{ik} \int \left(\varphi \left((\Omega^P)^* \right)_{kj}[1] - \left((\Omega^P)^* \right)_{kj}[\varphi] \right) \left(\overline{\Omega^P} \right)_{ij} \chi_{D_\Lambda} \\ &\stackrel{\text{supp } \varphi}{=} \omega_{ik} \int \varphi \left((\Omega^P)^* \right)_{kj}[1] \left(\overline{\Omega^P} \right)_{ij} - \omega_{ik} \int \left(\varphi \left((\Omega^P)^* \right)_{kj}[1] - \left((\Omega^P)^* \right)_{kj}[\varphi] \right) \left(\overline{\Omega^P} \right)_{ij} \chi_{D_\Lambda} \\ &\stackrel{(5.14)}{=} 0 - \omega_{ik} \int \left(\varphi \left((\Omega^P)^* \right)_{kj}[1] - \left((\Omega^P)^* \right)_{kj}[\varphi] \right) \left(\overline{\Omega^P} \right)_{ij} \chi_{D_\Lambda} \\ &= \omega_{ik} \int \mathcal{C}(\varphi, \Omega_{kj}^P)[\left(\overline{\Omega^P} \right)_{ij} \chi_{D_\Lambda}], \end{aligned}$$

where we denote the commutator \mathcal{C}

$$\mathcal{C}(b, T)[f] := b T f - T(bf).$$

Thus, we arrive at

$$\begin{aligned} - \int \omega : so(\Omega^P[|\nabla|^\mu \varphi])_{ij} &= \omega_{ik} \int \mathcal{C}(\varphi, (\Omega^P)_{kj}) [(\overline{\Omega^P})_{ij} \chi_{D_\Lambda}] \\ &\quad + \int \omega H_\mu(\varphi, P - I) P^T : \overline{\Omega^P} \chi_{D_\Lambda} \\ &\quad + \int \omega : \Omega^P [(1 - \chi_{D_\Lambda}) |\nabla|^\mu \varphi]. \end{aligned}$$

One checks, that

$$\mathcal{C}(\varphi, (\Omega^P)_{kj}) [(\overline{\Omega^P})_{ij} \chi_{D_\Lambda}] = P_{kl} \mathcal{C}(\varphi, \Omega^{ls}) [P_{sj}^T (\overline{\Omega^P})_{ij} \chi_{D_\Lambda}]$$

Next, (and here the antisymmetry of Ω , (5.9), plays its role)

$$\begin{aligned} so(\Omega^P[|\nabla|^\mu \varphi])_{ij} &= so(|\nabla|^\mu (P - I) P^T)_{ij} |\nabla|^\mu \varphi + \frac{1}{2} P_{ik} \Omega_{kl} [P_{jl} |\nabla|^\mu \varphi] - \frac{1}{2} P_{jk} \Omega_{kl} [P_{il} |\nabla|^\mu \varphi] \\ &\stackrel{(5.9)}{=} so(|\nabla|^\mu (P - I) P^T)_{ij} |\nabla|^\mu \varphi + \frac{1}{2} P_{ik} \Omega_{kl} [P_{jl} |\nabla|^\mu \varphi] + \frac{1}{2} P_{jl} \Omega_{kl} [P_{ik} |\nabla|^\mu \varphi] \\ &= so(|\nabla|^\mu (P - I) P^T)_{ij} |\nabla|^\mu \varphi + \frac{1}{2} P_{ik} \Omega_{kl} [P_{jl} |\nabla|^\mu \varphi] + \frac{1}{2} P_{jl} P_{ik} \Omega_{kl} [|\nabla|^\mu \varphi] \\ &\quad - \frac{1}{2} P_{jl} \mathcal{C}(P_{ik}, \Omega_{kl}) [|\nabla|^\mu \varphi] \\ &= so(|\nabla|^\mu (P - I) P^T)_{ij} |\nabla|^\mu \varphi + P_{ik} \Omega_{kl} [P_{jl} |\nabla|^\mu \varphi] \\ &\quad + \frac{1}{2} P_{ik} \mathcal{C}(P_{jl}, \Omega_{kl}) [|\nabla|^\mu \varphi] - \frac{1}{2} P_{jl} \mathcal{C}(P_{ik}, \Omega_{kl}) [|\nabla|^\mu \varphi] + \int \Omega^P [(1 - \chi_{D_\Lambda}) |\nabla|^\mu \varphi]. \end{aligned}$$

and

$$\begin{aligned} so(|\nabla|^\mu (P - I) P^T) &= \frac{1}{2} |\nabla|^\mu (P - I) P^T - \frac{1}{2} P |\nabla|^\mu (P^T - I) \\ &= |\nabla|^\mu (P - I) P^T + \frac{1}{2} (-|\nabla|^\mu (P - I) P^T - P |\nabla|^\mu (P^T - I)) \\ &= |\nabla|^\mu (P - I) P^T + \frac{1}{2} (|\nabla|^\mu (P P^T) - |\nabla|^\mu (P - I) P^T - P |\nabla|^\mu (P^T - I)) \\ &= |\nabla|^\mu (P - I) P^T + \frac{1}{2} H_\mu(P - I, P^T - I) \end{aligned}$$

This implies finally (going with $\omega_{ij} \in \{-1, 0, 1\}$ through all the possible matrices with two non-zero entries)

$$\begin{aligned} - \int \Omega^P [|\nabla|^\mu \varphi] &= \frac{1}{2} \int H_\mu(P - I, P^T - I) |\nabla|^\mu \varphi \\ &\quad + \int so(PC(\varphi, \Omega) [P^T \overline{\Omega^P}^T \chi_{D_\Lambda}]) \\ &\quad + \int so(\overline{\Omega^P} \chi_{D_\Lambda} P H_\mu(\varphi, P^T - I)) \\ &\quad - \int so(\mathcal{C}(P, \Omega) [|\nabla|^\mu \varphi] P^T) \\ &\quad + \int \Omega^P [(1 - \chi_{D_\Lambda}) |\nabla|^\mu \varphi]. \end{aligned}$$

□

Then, using the commutator estimates in [CRW76], (4.16), (4.17), and (4.18), we have shown the following Lemma, which implies Theorem 1.6

Lemma 5.7. *Let P be a minimizer of $E(\cdot)$ as in (5.6), Lemma 5.6. Assume moreover, that Ω satisfies (1.6). Then for any $\varphi \in C_0^\infty(B_r)$*

$$-\int \Omega^P [|\nabla|^\mu \varphi] \lesssim \Lambda^{-\frac{m}{2}-\mu} r^{\frac{m}{2}-\mu} \|A\|_2 [\varphi]_{BMO} + \|A\|_2^2 \begin{cases} [\varphi]_{BMO} & \text{if } \mu \in (0, 1], \\ \| |\nabla|^\mu \varphi \|_{(2,\infty)} & \text{if } \mu > 1. \end{cases}$$

Proof. By Lemma 5.5 and Lemma 5.6,

$$\|\Omega^P\|_{2 \rightarrow 1} + \|\overline{\Omega^P}\|_2 + \| |\nabla|^\mu P \|_2 \lesssim \|\Omega\|_{2 \rightarrow 1} \lesssim \|A\|_2,$$

and by Lemma 5.6 we need to estimate

$$\int H_\mu(P - I, P^T - I) |\nabla|^\mu \varphi \tag{5.15}$$

$$|\int so(PC(\varphi, \Omega) [P^T \overline{\Omega^P}^T \chi_{D_\Lambda}])| \lesssim \|A\|_2 \|C(\varphi, \mathcal{R}) [P^T \overline{\Omega^P}^T \chi_{D_\Lambda}]\|_2, \tag{5.16}$$

$$|\int so(\overline{\Omega^P} \chi_{D_\Lambda} P H_\mu(\varphi, P^T - I))| \lesssim \|\overline{\Omega^P}\|_2 \|H_\mu(\varphi, P^T - I)\|_2, \tag{5.17}$$

$$|\int so(C(P, \Omega) [|\nabla|^\mu \varphi] P^T)| \lesssim \|A\|_2 \|C(P, \mathcal{R}) [|\nabla|^\mu \varphi]\|_2, \tag{5.18}$$

$$|\int \Omega^P [(1 - \chi_{D_\Lambda}) |\nabla|^\mu \varphi]| \lesssim \|\Omega^P\|_{2 \rightarrow 1} \|(1 - \chi_{D_\Lambda}) |\nabla|^\mu \varphi\|_2. \tag{5.19}$$

The estimate of (5.15) is immediate from (4.18), for the estimate of (5.16) we apply [CRW76]. For the estimate of (5.17) we use (4.17), for (5.18) we have (4.16).

It remains to estimate (5.19), which follows from

$$\begin{aligned} \|(1 - \chi_{D_\Lambda}) |\nabla|^\mu \varphi\|_2 &\lesssim \sum_{k=1}^{\infty} \| |\nabla|^\mu \varphi \|_{2, A_{\Lambda r}^k} \stackrel{L.B.1}{\lesssim} \sum_{k=1}^{\infty} (2^k \Lambda r)^{-\frac{m}{2}-\mu} \|\varphi\|_1 \\ &\stackrel{P.5.4}{\lesssim} \sum_{k=1}^{\infty} (2^k \Lambda r)^{-\frac{m}{2}-\mu} r^m [\varphi]_{BMO}. \end{aligned}$$

□

A. Some facts on our fractional operators

The fractional laplacian $\Delta^{\frac{s}{2}}$ is usually defined via its Fourier-symbol $-|\xi|^s$. Here, we will mostly use the negative fractional laplacian $(-\Delta)^{\frac{s}{2}} \equiv |\nabla|^s$ (which here plays the role of the gradient, or the divergence and rotation in the classical settings), defined via its symbol $|\xi|^s$. These operators are defined for $s \in (-m, m)$, if $s < 0$, we write $\Delta^{\frac{s}{2}} \equiv I_{|s|}$.

Most of the time, we will use the potential definition: For Schwartz functions f ,

$$|\nabla|^s f(x) = c_s \lim_{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} \frac{f(x) - f(y)}{|x-y|^{m+s}} dy \quad \text{for } s \in (0, 2).$$

The inverse is the Riesz potential,

$$I_s f(x) = \tilde{c}_s \lim_{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{|x-y|^{m-s}} dy \quad \text{for } s \in (0, m).$$

We refer, e.g., to [SKM93, Lan72] on hyper-singular operators, generalizations, and different representation formulas.

Next, we state some useful facts about the fractional laplacian, which we are going to use throughout our paper, as standard repertoire.

We have the standard Poincaré inequality, for a proof, we refer, e.g., to [Sch10b].

Lemma A.1 (Poincaré inequality with compact support). *Let $s \in [0, m)$, $p \in (1, \infty)$, $q \in [1, \infty]$, then for any $B_r \subset \mathbb{R}^m$, and any $f \in C_0^\infty(B_r)$*

$$\|f\|_{(p_1, q_1)} \leq C_s r^s \| |\nabla|^s f \|_{(p_1, q_1)}.$$

The (scaling invariant) Sobolev inequality takes the form

Lemma A.2 (Sobolev inequality). *Let $s \in [0, m)$, $p_1, p_2 \in [1, \infty)$, $q \in [1, \infty]$, for any $f \in \mathcal{S}(\mathbb{R}^m)$,*

$$\|f\|_{(p_1, q)} \leq \| |\nabla|^s f \|_{(p_2, q)},$$

where

$$\frac{1}{p_2} = \frac{1}{p_1} + \frac{s}{m}.$$

For $p_1 = \infty$, we have the following limiting version of Sobolev's inequality:

Lemma A.3 (Limiting Sobolev inequality). *Let $s \in (0, m)$. For any $f \in \mathcal{S}(\mathbb{R}^m)$,*

$$\|f\|_\infty \leq \| |\nabla|^s f \|_{(\frac{m}{s}, 1)},$$

Also, we have the following Hölder-like inequality

Lemma A.4 (Hölder inequality). *Let $s \in [0, m)$, then for any $p_1 < p_2$, for any $B_r \subset \mathbb{R}^m$, and any $f \in C_0^\infty(B_r)$*

$$\| |\nabla|^s f \|_{(p_1, q_1)} \leq C_{s, p_1, p_2} r^{\frac{m}{p_1} - \frac{m}{p_2}} \| |\nabla|^s f \|_{(p_2, \infty)}$$

Proof. Let $\Lambda > 2$, then

$$\| |\nabla|^s f \|_{(p_1, q_1), B_{\Lambda r}} \lesssim C_{s, p_1, p_2, \Lambda} r^{\frac{m}{p_1} - \frac{m}{p_2}} \| |\nabla|^s f \|_{(p_2, \infty)}.$$

On the other hand, for some $\theta > 0$, by Lemma B.1, Lemma A.1,

$$\| |\nabla|^s f \|_{(p_1, q_1), \mathbb{R}^n \setminus B_{\Lambda r}} \lesssim \Lambda^{-\theta} r^{-s} \|f\|_{(p_1, q_1)} \lesssim \Lambda^{-\theta} \| |\nabla|^s f \|_{(p_1, q_1)}.$$

For sufficiently large Λ we can absorb the latter term into the left-hand side, and obtain the claim. \square

From the Lemmata before, we also have

Lemma A.5 (Poincaré-Sobolev inequality with compact support). *Let $s \in (0, m)$, $p_1, q_1 \in (1, \infty)$, then we have $s \leq t$, for any $B_r \subset \mathbb{R}^m$, and any $f \in C_0^\infty(B_r)$*

$$\| |\nabla|^s f \|_{(p_1, q_1)} \leq C_{p_1, q_1, p_2, q_2, s, t} r^{\frac{m}{p_1} - \frac{m}{p_2} + s - t} \| |\nabla|^t f \|_{(p_2, q_2)},$$

where $p_2 \in (1, \infty)$ such that

$$\frac{1}{p_2} \leq \frac{1}{p_1} + \frac{s - t}{m}$$

and $q_2 = \infty$ if the above inequality is strict, else $q_1 = q_2$.

A very important ingredient in our arguments is the boundedness of the Riesz potential on Morrey spaces.

Lemma A.6 ([Ada75]). *Let $s \in [0, m)$, $p_1, p_2 \in (1, \infty)$, $q \in [1, \infty]$, $\lambda \in (0, m]$, such that*

$$\frac{1}{p_1} = \frac{1}{p_2} - \frac{s}{\lambda}.$$

Then for any $f \in \mathcal{S}(\mathbb{R}^m)$,

$$\|I_s f\|_{(p_1, q)_\lambda} \lesssim \|I_s f\|_{(p_2, q)_\lambda}.$$

The following is an easy equivalence result, recall (1.17).

Proposition A.7. *Let $\Lambda > 2$, $\sigma > 0$. Then,*

$$\sum_{k=K_0}^{\infty} 2^{-k\sigma} \|f\|_{(p, q), A_r^k} \leq C_\sigma \|f\|_{(p, q), B_{\Lambda r}} + \sum_{k=0}^{\infty} 2^{-k\sigma} \|f\|_{(p, q), A_{\Lambda r}^k}$$

Proof. Let $k_0 := \lceil \log_2 \Lambda \rceil \geq 1$, then

$$2^{k_0} \leq \Lambda < 2^{k_0+1}$$

We have

$$2^{-\sigma(l+k_0)} \|f\|_{(p, q), A_{2^{l+k_0}r}^k} \lesssim 2^{-\sigma k_0} 2^{-\sigma(l-1)} \|f\|_{(p, q), A_{2^{l-1}\Lambda r}^k} + 2^{-\sigma k_0} 2^{\sigma l} \|f\|_{(p, q), A_{2^l \Lambda r}^k}.$$

□

B. Quasi-locality

In this section we gather some facts which quantify the quasi-local behaviour of operators like fractional laplacians $|\nabla|^\alpha$, Riesz transforms \mathcal{R} , and Riesz potentials I_s . With “quasi-local” we mean the following: Let $A \subset \mathbb{R}^m$ be some domain and assume that $\text{supp } f \subset A$. If we take T to be any of the above mentioned operators, then there is no reason why $\text{supp } Tf \subset A$, nor $\text{supp } Tf \subset B_\delta A$ for some $\delta > 0$. Nevertheless, if we take a domain $B \subset \mathbb{R}^m$, $\text{dist}(A, B) > \epsilon > 0$, then $Tf \in C^\infty(B)$. In fact, in this case

$$Tf(x) = k * f(x) \quad \text{for } x \in B,$$

where k is a kernel of the form $k(y) = h(y/|y|) |y|^{-n-s}$ for some $s \in (-m, m)$, h some smooth function on \mathbb{S}^{m-1} . Since $\text{supp } f \subset A$ and $x \in B$, we can replace

$$Tf(x) = \tilde{k} * f(x),$$

where $\tilde{k}(y) = (1 - \eta(y))k(y)$, and $\eta \in C_0^\infty(B_\epsilon(0))$, $\eta \equiv 1$ on $B_{\epsilon/2}(0)$. Obviously, $\tilde{k} \in C^\infty(\mathbb{R}^m)$, and consequently so is Tf . In fact, by the usual Young-inequality, we have

$$\|Tf\|_{\infty, B} \leq \|\tilde{k}\|_\infty \|f\|_1 \leq \|k\|_{\infty, \mathbb{R}^m \setminus B_{\epsilon/2}(0)} \|f\|_1 \leq C_{\|h\|_\infty} \epsilon^{-n-s} \|f\|_1.$$

That is, although we cannot ensure that $Tf \equiv 0$ in B (as it would be, e.g., the case for local operators like ∇), we can at least quantify that the farther away B is from A , the less is the norm of Tf on B . In particular, we have

Lemma B.1 (Quasi-locality (I)). *Let $p_1, p_2, q_1, q_2 \in [1, \infty]$, $s \in (-m, m)$ and $\Omega_1, \Omega_2 \subset \mathbb{R}^m$ be disjoint domains with $d := \text{dist}(\Omega_1, \Omega_2) > 0$ and with positive and finite Lebesgue measure. Then, for any $f \in \mathcal{S}(\mathbb{R})$,*

$$\|\Delta^{\frac{s}{2}}(f\chi_{\Omega_2})\|_{(p_1, q_1), \Omega_1} \leq d^{-m-s} |\Omega_1|^{1/p_1} |\Omega_2|^{1-1/p_2} \|f\|_{(p_2, q_2), \Omega_2},$$

where we set

$$\Delta^{\frac{s}{2}} := \begin{cases} |\nabla|^s & \text{if } s > 0, \\ Id \text{ or } \mathcal{R} & \text{if } s = 0, \\ I_{|s|} & \text{if } s < 0. \end{cases}$$

Often we will use the above also for Ω_1 or Ω_2 to be a complement of some ball B_r . This is valid, since $\mathbb{R}^m \setminus B_r = \bigcup_{k=1}^\infty A_r^k$, recall (1.17). Then

$$\chi_{\mathbb{R}^m \setminus B_r} = \sum_{k=1}^\infty \chi_{A_r^k},$$

and for each A_r^k we have the correct estimate, so that for $s \in (-m, m)$ the sum on k is convergent. Consequently, as a special case, using also Poincaré inequality (cf. Section A), we have

Corollary B.2 (Quasi-locality (II)). *Let $p_1, p_2 \in (1, \infty)$, $q_1, q_2 \in [1, \infty]$, $s, t \in [0, m)$. Then, for any $B_r \subset \mathbb{R}^m$, $f \in \mathcal{S}(\mathbb{R})$, $\Lambda > 1$, whenever*

$$\| |\nabla|^s (f\chi_{B_r}) \|_{(p_1, q_1), \mathbb{R}^m \setminus B_{\Lambda r}} \leq C_{s, p_1, p_2, q_1} \Lambda^{-m-s+\frac{m}{p_1}} r^{\frac{m}{p_1} - \frac{m}{p_2} - s+t} \| |\nabla|^t (\chi_{B_r} f) \|_{(p_2, q_2), B_r}.$$

Lemma B.3 (Quasilocality (III)). *Let $f, g \in \mathcal{S}(\mathbb{R}^m)$, $\Omega_1, \Omega_2 \subset \mathbb{R}^m$ be disjoint domains with $d := \text{dist}(\Omega_1, \Omega_2) > 0$ and with positive and finite Lebesgue measure.*

$$\| |\nabla|^s ((\Delta^{\frac{t}{2}} f\chi_{\Omega_1})g\chi_{\Omega_2}) \|_{(p_1, q_1)} \lesssim \sup_{\alpha \in [0, s]} d^{-m-t-\alpha} \|f\chi_{\Omega_1}\|_1 \| |\nabla|^{s-\alpha} (g\chi_{\Omega_2}) \|_{(p_1, q_1)}$$

for any $t \in (-m, m)$, $s \in (0, m)$.

Proof. By the disjoint support, pointwise almost everywhere,

$$(\Delta^{\frac{t}{2}} f\chi_{\Omega_1}) g\chi_{\Omega_2}(x) = c \int |x-y|^{-m-t} f(y)\chi_{\Omega_1}(y) g(x)\chi_{\Omega_2}(x) dy.$$

Let $\eta \equiv \eta \in C^\infty([0, \infty))$, $\eta \equiv 1$ on $[1, \infty)$, $\eta \equiv 0$ on $[0, 1]$, and set

$$K_{t,d}(z) := |z|^{-m-t} \eta(z/d) \in L^p(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m) \cap C^\infty(\mathbb{R}^m) \quad \text{for any } p > \max\{\frac{m}{m+t}, 1\}.$$

It is worth noting the scaling behaviour of $K_{t,d}$ in d ,

$$K_{t,d}(z) = d^{-m-t} K_{t,1}(z/d). \quad (\text{B.1})$$

We have

$$(\Delta^{\frac{t}{2}} f \chi_{\Omega_1}) g \chi_{\Omega_2} = (K_{t,d} * f \chi_{\Omega_1}) g \chi_{\Omega_2}.$$

Now we know that

$$|\nabla|^s K_{t,d} \in C^\infty(\mathbb{R}^m),$$

and on the other hand

$$\| |\nabla|^s K_{t,d}(x) \| \lesssim C_d |x|^{-m-s-t},$$

so together

$$|\nabla|^s K_{t,d} \in L^p(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m) \cap C^\infty(\mathbb{R}^m), \quad \text{for any } p > \max\left\{\frac{m}{m+t+s}, 1\right\}.$$

By the scaling (B.1), we also know the exact dependence on d :

$$\| |\nabla|^\alpha K_{t,d} \|_{(p,q)} = d^{-m-t-\alpha+\frac{m}{p}} \| |\nabla|^\alpha K_{t,1} \|_{(p,q)}$$

Consequently,

$$\| |\nabla|^s (K_{t,d} * f \chi_{\Omega_1}) \|_\infty \lesssim \| f \chi_{\Omega_1} \|_1.$$

Then our product rule, Lemma 4.2, tells us that we essentially have to estimate

$$\begin{aligned} & \| |\nabla|^s ((|\nabla|^t f \chi_{\Omega_1}) g \chi_{\Omega_2}) \|_{(p_1, q_1)} \\ & \lesssim \sup_{\alpha \in [0, s]} \| |\nabla|^\alpha K_{t,d} * f \chi_{\Omega_1} \| \| |\nabla|^{s-\alpha} (g \chi_{\Omega_2}) \|_{(p_1, q_1)} \\ & \lesssim \sup_{\alpha \in [0, s]} \| |\nabla|^\alpha K_{t,d} * f \chi_{\Omega_1} \|_\infty \| |\nabla|^{s-\alpha} (g \chi_{\Omega_2}) \|_{(p_1, q_1)} \\ & \lesssim \sup_{\alpha \in [0, s]} \| |\nabla|^\alpha K_{t,d} \|_\infty \| f \chi_{\Omega_1} \|_1 \| |\nabla|^{s-\alpha} (g \chi_{\Omega_2}) \|_{(p_1, q_1)} \\ & \lesssim \sup_{\alpha \in [0, s]} d^{-m-t-\alpha} \| f \chi_{\Omega_1} \|_1 \| |\nabla|^{s-\alpha} (g \chi_{\Omega_2}) \|_{(p_1, q_1)}. \end{aligned}$$

□

C. Left-hand side estimates

Lemma C.1 (Left-hand side estimates). *For a uniform constant C , and any $\kappa \in [\mu, 2\mu)$, $\mu \leq \frac{m}{2}$, $\Lambda \geq 4$,*

$$\begin{aligned} \|v\|_{(\frac{m}{m-\kappa}, \infty), B_{\Lambda^{-1}r}} &\leq C \sup_{\varphi \in C_0^\infty(B_r, \mathbb{R}^N)} \frac{1}{\| |\nabla|^\kappa \varphi \|_{(\frac{m}{\tau+\kappa-\mu}, 1)}} \int v \cdot |\nabla|^\mu \varphi \\ &+ C \Lambda^{\kappa-m} \|v\|_{(\frac{m}{m-\kappa}, \infty), B_r} \\ &+ C \Lambda^{\kappa-m} \sum_{k=0}^{\infty} 2^{k(\kappa-m)} \|v\|_{(\frac{m}{m-\kappa}, \infty), B_{A_r^k}}. \end{aligned}$$

More generally, for $\alpha \in (0, \mu]$,

$$\begin{aligned} \| |\nabla|^{\mu-\alpha} v \|_{(\frac{m}{m+\mu-\alpha-\kappa}, \infty), B_{\Lambda^{-1}r}} &\leq C \sup_{\varphi \in C_0^\infty(B_r, \mathbb{R}^N)} \frac{1}{\| |\nabla|^\alpha \varphi \|_{(\frac{m}{\alpha+\kappa-\mu}, 1)}} \int v \cdot |\nabla|^\mu \varphi \\ &+ C \Lambda^{\kappa-m+\alpha-\mu} \|v\|_{(\frac{m}{m-\kappa}, \infty), B_r} \\ &+ C \Lambda^{\kappa-m+\alpha-\mu} \sum_{k=0}^{\infty} 2^{k(\kappa-m+\alpha-\mu)} \|v\|_{(\frac{m}{m-\kappa}, \infty), A_r^k}. \end{aligned}$$

Proof. Let $f \in C_0^\infty(B_{\Lambda^{-1}r}, \mathbb{R}^N)$, $\|f\|_{(\frac{m}{\alpha+\kappa-\mu}, 1)} \leq 1$ such that

$$\| |\nabla|^{\mu-\alpha} v \|_{(\frac{m}{m+\mu-\alpha-\kappa}, \infty), B_{\Lambda^{-1}r}} \leq 2 \int |\nabla|^{\mu-\alpha} v \cdot f.$$

Decompose for the usual cutoff $\eta_{r/2} \in C_0^\infty(B_{\frac{r}{2}})$,

$$f = |\nabla|^\alpha (\eta_{\frac{r}{2}} I_\alpha f) + |\nabla|^\alpha ((1 - \eta_{\frac{r}{2}}) I_\alpha f) =: |\nabla|^\alpha g_1 + |\nabla|^\alpha g_2.$$

As usual, using Lemma 4.1 (for $\beta = 0$) as an approximate product rule, for finitely many $s_k \in [0, \alpha]$,

$$\| |\nabla|^\alpha g_1 \|_{(\frac{m}{\alpha+\kappa-\mu}, 1)} \lesssim \sum_k \|I_{s_k} | |\nabla|^\alpha \eta_{\frac{r}{2}} | \|_{(\frac{m}{\alpha-s_k}, \infty)} \|I_{\alpha-s_k} |f| \|_{(\frac{m}{s_k+\kappa-\mu}, 1)} \lesssim \|f\|_p.$$

As for g_2 , for a usual decomposition unity $\eta_l \in C_0^\infty(B_{2^l r} \setminus B_{2^{l-2} r})$

$$|\nabla|^\alpha g_2 = \sum_{l=-2}^{\infty} |\nabla|^\alpha (\eta_l I_\alpha f) =: \sum_{l=-2}^{\infty} |\nabla|^\alpha \tilde{g}_l$$

and with the help of Lemma B.3,

$$\| |\nabla|^\mu \tilde{g}_l \|_{(\frac{m}{\kappa}, 1)} \lesssim (2^l \Lambda)^{\kappa-m-\mu+\alpha} \|f\|_{\frac{m}{\alpha+\kappa-\mu}} \leq (2^l \Lambda)^{\kappa-m+\alpha-\mu} \|f\|_{\frac{m}{\alpha+\kappa-\mu}},$$

and for $k \geq 1$,

$$\| |\nabla|^\mu \tilde{g}_l \|_{(\frac{m}{\kappa}, 1), B_{2^k r} \setminus B_{2^{k-1} r}} \lesssim \Lambda^{\kappa-m+\alpha-\mu} 2^{k\kappa+\max\{k,l\}(-m-\mu)+l\alpha} \|f\|_{\frac{m}{\alpha+\kappa-\mu}}.$$

Consequently, for any $k \in \mathbb{N}_0$,

$$\| |\nabla|^\mu g_2 \|_{(\frac{m}{\kappa}, 1), A_r^k} \lesssim (2^k \Lambda)^{\kappa-m+\alpha-\mu} \|f\|_{\frac{m}{\alpha+\kappa-\mu}}.$$

So we conclude using

$$\int v \cdot |\nabla|^\mu g_2 \lesssim \|v\|_{(\frac{m}{m-\kappa}, \infty), B_r} \| |\nabla|^\mu g_2 \|_{(\frac{m}{\kappa}, 1), B_r} + \sum_{k=1}^{\infty} \|v\|_{(\frac{m}{m-\kappa}, \infty), A_r^k} \| |\nabla|^\mu g_2 \|_{(\frac{m}{\kappa}, 1), A_r^k}$$

□

D. Iteration

Lemma D.1. *Let $q \in (0, 1)$, $K > 1$, $\varepsilon > 0$ and assume*

$$\varepsilon + q^{2K} + K\varepsilon \frac{1}{1-q} \leq \frac{1}{16}. \quad (\text{D.1})$$

Let moreover $\psi, \Phi : (0, \infty) \rightarrow (0, \infty)$, $\psi \leq \Phi$, and Φ monotone rising. Assume that for all $\lambda \in (0, 1]$

$$\Phi(2^{-K}\lambda) \leq \varepsilon \Phi(\lambda) + \varepsilon \sum_{k=0}^{\infty} q^k \psi(2^k \lambda). \quad (\text{D.2})$$

If there is $G < \infty$ so that for all $\lambda \in (0, 1)$

$$\sum_{k=0}^{\infty} q^k \psi(2^k \lambda) \leq G.$$

Then, for all $\lambda \in (0, 1)$,

$$\Phi(\lambda) \leq 32 \lambda^{\frac{1}{2K}} (\Phi(1) + G).$$

For a proof, we refer, e.g., to the arxiv-version of [Sch11], or [BRS12].

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