

HOMOTOPY GROUPS OF SPHERES AND LIPSCHITZ HOMOTOPY GROUPS OF HEISENBERG GROUPS

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ABSTRACT. We provide a sufficient condition for the nontriviality of the Lipschitz homotopy group of the Heisenberg group, $\pi_m^{\text{Lip}}(\mathbb{H}_n)$, in terms of properties of the classical homotopy group of the sphere, $\pi_m(\mathbb{S}^n)$. As an application we provide a new simplified proof of the fact that $\pi_n^{\text{Lip}}(\mathbb{H}_n) \neq \{0\}$, $n = 1, 2, \dots$ and we prove a new result that $\pi_{4n-1}^{\text{Lip}}(\mathbb{H}_{2n}) \neq \{0\}$ for $n = 1, 2, \dots$. The last result is based on a new generalization of the Hopf invariant. We also prove that Lipschitz mappings are not dense in the Sobolev space $W^{1,p}(\mathcal{M}, \mathbb{H}_{2n})$ when $\dim \mathcal{M} \geq 4n$ and $4n - 1 \leq p < 4n$.

1. INTRODUCTION

In this paper, we provide further evidence for the role of Lipschitz homotopy groups in the development of analysis on (non-Riemannian) metric spaces, and specifically, in the study of Sobolev mappings with non-Riemannian target spaces such as the sub-Riemannian Heisenberg group. We link the study of Lipschitz homotopy groups of Heisenberg groups with classical homotopy theory through a new notion of *rank-essential* homotopy groups (Definition 1.4). Using this approach, we provide new and simplified proofs of the nontriviality of certain Lipschitz homotopy groups of Heisenberg groups (previously established in [8]) as well as new examples of nontrivial Lipschitz homotopy groups. These results have applications to the problem of density of Lipschitz mappings in Sobolev spaces with Heisenberg targets.

The Heisenberg group \mathbb{H}_n is \mathbb{R}^{2n+1} equipped with the so called Carnot-Carathéodory metric d_{cc} . For every compact set K there is a constant $C \geq 1$ such that $C^{-1}|x - y| \leq d_{cc}(x, y) \leq C|x - y|^{1/2}$ for $x, y \in K$. Thus \mathbb{H}_n is homeomorphic to \mathbb{R}^{2n+1} and the identity mapping

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$\text{id} : \mathbb{H}_n \rightarrow \mathbb{R}^{2n+1}$ is locally Lipschitz. However, the inverse mapping $\text{id} : \mathbb{R}^{2n+1} \rightarrow \mathbb{H}_n$ is only locally Hölder continuous with exponent $1/2$. There is no bi-Lipschitz homeomorphism between \mathbb{H}_n and \mathbb{R}^{2n+1} , because the Hausdorff dimension of every open set in \mathbb{H}_n is $2n + 2$. The following result is well known.

Proposition 1.1. *If $f : \mathbb{R}^k \supset \Omega \rightarrow \mathbb{H}_n$ is Lipschitz continuous, where Ω is open, then it is locally Lipschitz continuous as a mapping into \mathbb{R}^{2n+1} . Hence f is differentiable a.e. It turns out that $\text{rank } df \leq n$ a.e.*

Since \mathbb{H}_n is homeomorphic to \mathbb{R}^{2n+1} , all of its homotopy groups are trivial. On the other hand the Heisenberg group, as an object of study from the viewpoint of geometric analysis on metric spaces, is naturally equipped with its Carnot-Carathéodory metric d_{cc} (or other metrics bi-Lipschitz equivalent to d_{cc}). As observed above, the Euclidean metric is not of this type. In the framework of analysis on metric spaces it is natural to consider *Lipschitz homotopy groups*, which are only insensitive to bi-Lipschitz deformation. The Lipschitz homotopy groups $\pi_n^{\text{Lip}}(X)$ of a metric space X are defined in the same way as the classical homotopy groups with the difference that now both mappings and homotopies between them are required to be Lipschitz.

In the case of Riemannian manifolds homotopy groups and Lipschitz homotopy groups are the same since continuous mappings can be smoothly approximated. However for non-smooth spaces they may differ. The Heisenberg group is an example since its n th Lipschitz homotopy group $\pi_n^{\text{Lip}}(\mathbb{H}_n) \neq \{0\}$ is non-trivial, [1, 8]. However, $\pi_m^{\text{Lip}}(\mathbb{H}_n) = \{0\}$ for all $1 \leq m < n$ [8, 27] and $\pi_m^{\text{Lip}}(\mathbb{H}_1) = \{0\}$ for all $m \geq 2$ [28]. These results show an analogy between the Lipschitz homotopy groups of \mathbb{H}_n and the homotopy groups of the sphere \mathbb{S}^n . The nontriviality of $\pi_n^{\text{Lip}}(\mathbb{H}_n)$ is based on the following fact (see [1, Section 4], [8, Theorem 3.2], [9, Example 3.1]).

Proposition 1.2. *There is a bi-Lipschitz embedding of the sphere $\phi : \mathbb{S}^n \rightarrow \mathbb{H}_n$ which is a smooth embedding as a mapping to \mathbb{R}^{2n+1} .*

It was proved in [1, 8] that such an embedding cannot be extended to a Lipschitz map $\Phi : \mathbb{B}^{n+1} \rightarrow \mathbb{H}_n$. Another simpler proof of this fact is provided below. See the proof that $\pi_n(\mathbb{S}^n)$ is rank-essential later in this section. Thus $\pi_n^{\text{Lip}}(\mathbb{H}_n) \neq \{0\}$. To emphasize the analogy between $\pi_m(\mathbb{S}^n)$ and $\pi_m^{\text{Lip}}(\mathbb{H}_n)$ it was asked in [8, Question 4.16] whether any bi-Lipschitz embedding $\phi : \mathbb{S}^n \rightarrow \mathbb{H}_n$ induces an injective homomorphism $\pi_m(\mathbb{S}^n) \rightarrow \pi_m^{\text{Lip}}(\mathbb{H}_n)$. Actually the authors of the question expected

that if a smooth mapping $f : \mathbb{S}^m \rightarrow \mathbb{S}^n$ is not homotopic to a constant map $0 \neq [f] \in \pi_m(\mathbb{S}^n)$, then the map $g = \phi \circ f : \mathbb{S}^m \rightarrow \mathbb{H}_n$ cannot be extended to a Lipschitz map $G : \mathbb{B}^{m+1} \rightarrow \mathbb{H}_n$. As it will be explained below there were strong reasons based on the Sard theorem to believe in this conjecture, but surprisingly the conjecture is false! Namely, recently Wenger and Young [28, Theorem 1] proved the following result.

Theorem 1.3. *If $\alpha : \mathbb{S}^n \rightarrow \mathbb{H}_n$ and $\beta : \mathbb{S}^m \rightarrow \mathbb{S}^n$ are Lipschitz and $n+2 \leq m < 2n-1$, then the map $g = \alpha \circ \beta : \mathbb{S}^m \rightarrow \mathbb{H}_n$ can be extended to a Lipschitz map $G : \mathbb{B}^{m+1} \rightarrow \mathbb{H}_n$.*

In particular $\pi_7(\mathbb{S}^5) = \mathbb{Z}_2$, so there is a smooth map $f : \mathbb{S}^7 \rightarrow \mathbb{S}^5$ that is not homotopic to a constant map, but if $\phi : \mathbb{S}^5 \rightarrow \mathbb{H}_5$ is a bi-Lipschitz embedding, then $g = \phi \circ f : \mathbb{S}^7 \rightarrow \mathbb{H}_5$ has a Lipschitz extension $G : \mathbb{B}^8 \rightarrow \mathbb{H}_5$. This is just one example, but the above theorem leads to many more examples. It just suffices to look at the table of the homotopy groups of the spheres to find cases when $\pi_m(\mathbb{S}^n) \neq \{0\}$ and $n+2 \leq m < 2n-1$. It is important to note here that it does not necessarily imply that $\pi_m^{\text{Lip}}(\mathbb{H}_n) = \{0\}$, because in this construction we consider mappings to \mathbb{H}_n that factor through \mathbb{S}^n via a bi-Lipschitz embedding into \mathbb{H}_n . Perhaps there are other mappings from \mathbb{S}^m to \mathbb{H}_n that are not Lipschitz homotopic to constant mappings.

Definition 1.4. We say that the homotopy group $\pi_m(\mathbb{S}^n)$ is *rank-essential* if there is $f \in C^\infty(\mathbb{S}^m, \mathbb{S}^n)$ with the following property (R): for every Lipschitz extension $F : \mathbb{B}^{m+1} \rightarrow \mathbb{R}^{n+1}$, $F|_{\partial\mathbb{B}^{m+1}} = f$, we have

$$\text{rank } dF = n + 1$$

on a set of positive measure.

Clearly if $\pi_m(\mathbb{S}^n)$ is rank-essential, then $\pi_m(\mathbb{S}^n) \neq \{0\}$. The definition is motivated by the following result.

Theorem 1.5. *If $\pi_m(\mathbb{S}^n)$ is rank-essential, then $\pi_m^{\text{Lip}}(\mathbb{H}_n) \neq \{0\}$.*

Proof. Suppose by contrary that $\pi_m(\mathbb{S}^n)$ is rank-essential and that $\pi_m^{\text{Lip}}(\mathbb{H}_n) = \{0\}$. Let $f \in C^\infty(\mathbb{S}^m, \mathbb{S}^n)$ be a mapping with the property (R). Since $\pi_m^{\text{Lip}}(\mathbb{H}_n) = \{0\}$, $g = \phi \circ f : \mathbb{S}^m \rightarrow \mathbb{H}_n$ has a Lipschitz extension $G : \mathbb{B}^{m+1} \rightarrow \mathbb{H}_n$. Here $\phi : \mathbb{S}^n \rightarrow \mathbb{H}_n$ is a bi-Lipschitz embedding from Proposition 1.2. By Proposition 1.1 $\text{rank } dG \leq n$ a.e. where now we regard G as a mapping into \mathbb{R}^{2n+1} . The mapping $\phi^{-1} : \phi(\mathbb{S}^n) \rightarrow \mathbb{S}^n \subset \mathbb{R}^{n+1}$ is smooth and hence it admits a smooth extension $\Psi : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{n+1}$. Clearly $F = \Psi \circ G : \mathbb{B}^{m+1} \rightarrow \mathbb{R}^{n+1}$ is

a Lipschitz extension of $\Psi \circ G|_{\partial \mathbb{B}^{m+1}} = f$. Since $\text{rank } dG \leq n$ a.e. we also conclude that $\text{rank } dF = \text{rank } d(\Psi \circ G) \leq n$ which contradicts the property (R) of f . \square

As a corollary of Theorem 1.3 and the proof of Theorem 1.5 we obtain (see [28, Theorem 2] for a stronger statement where it is shown that the corollary is true also for $m = n + 1$).

Corollary 1.6. *If $n + 2 \leq m < 2n - 1$, then $\pi_m(\mathbb{S}^n)$ is not rank-essential.*

In particular if $n + 2 \leq m < 2n - 1$ and $\pi_m(\mathbb{S}^n) \neq \{0\}$ (for example $\pi_7(\mathbb{S}^5) = \mathbb{Z}_2$), then every smooth mapping $f : \mathbb{S}^m \rightarrow \mathbb{S}^n$, $[f] \neq 0$ admits a Lipschitz extension $F : \mathbb{B}^{m+1} \rightarrow \mathbb{R}^{n+1}$ with $\text{rank } dF \leq n$ a.e. despite the fact that the image of F contains the unit $(n + 1)$ -dimensional ball. Indeed, otherwise we could pick a point in $\mathbb{B}^{n+1} \setminus F(\mathbb{B}^{m+1})$ and retract F onto \mathbb{S}^n .

The main result of the paper reads as follows.

Theorem 1.7. *The homotopy groups $\pi_n(\mathbb{S}^n)$ and $\pi_{4n-1}(\mathbb{S}^{2n})$ are rank-essential for $n = 1, 2, 3, \dots$ and hence $\pi_n^{\text{Lip}}(\mathbb{H}_n) \neq \{0\}$, $\pi_{4n-1}^{\text{Lip}}(\mathbb{H}_{2n}) \neq \{0\}$.*

According to the Serre finiteness theorem [25] these are the only cases when the homotopy group of the sphere is infinite. The proof of Theorem 1.7 is based on differential forms. It is done explicitly in the case of $\pi_{4n-1}^{\text{Lip}}(\mathbb{H}_{2n})$ and implicitly in the case of $\pi_n^{\text{Lip}}(\mathbb{H}_n)$. In the latter case we use the fact that the ball cannot be retracted to the boundary which can be easily proved with the help of differential forms. The language of differential forms is useful when one wants to detect the rational homotopy groups of CW complexes $\pi_m(X) \otimes \mathbb{Q}$. This is the so-called rational homotopy theory discovered by Sullivan [12]. However in the case of spheres the rational homotopy groups $\pi_m(\mathbb{S}^n) \otimes \mathbb{Q}$ are nontrivial exactly in the cases covered by Theorem 1.7. This follows from the Serre finiteness theorem. It would be very interesting to see if $\pi_m(\mathbb{S}^n)$ is rank-essential for other values of m and n .

The fact that $\pi_n^{\text{Lip}}(\mathbb{H}_n) \neq \{0\}$ was proved in [1, 8], but the proof presented here is different and simpler since it does not refer to pure unrectifiability of the Heisenberg group, neither to the degree theory.

Proof that $\pi_n(\mathbb{S}^n)$ is rank-essential. Let $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$ be the identity map. If $F : \mathbb{B}^{n+1} \rightarrow \mathbb{R}^{n+1}$ is a Lipschitz extension, then $\mathbb{B}^{n+1} \subset$

$F(\mathbb{B}^{n+1})$. In particular the image of F has positive $(n+1)$ -dimensional measure. On the other hand for Lipschitz mappings we have ([11, Theorem 3.2.3])

$$|F(\mathbb{B}^{n+1})| \leq \int_{\mathbb{B}^{n+1}} |\det dF|.$$

Since $|F(\mathbb{B}^{n+1})| > 0$ we conclude that $\text{rank } dF \geq n+1$ on a set of positive measure. Thus $\pi_n(\mathbb{S}^n)$ is rank-essential and hence $\pi_n^{\text{Lip}}(\mathbb{H}_n) \neq \{0\}$. \square

In the last step of the proof we could refer to the Sard theorem for the Lipschitz mappings [23, Theorem 7.6] instead of the integral inequality used above. Assuming that $\text{rank } dF \leq n$ a.e. we would have that all points in \mathbb{B}^{n+1} are critical and hence the measure of the image $F(\mathbb{B}^{n+1})$ as the measure of the image of the critical set equals zero.

We now investigate the connection to the Sard theorem in greater detail.

Proposition 1.8. *Let $f \in C^\infty(\mathbb{S}^m, \mathbb{S}^n)$, $0 \neq [f] \in \pi_m(\mathbb{S}^n)$. Let*

$$F : \mathbb{B}^{m+1} \rightarrow \mathbb{R}^{n+1}, \quad F|_{\partial\mathbb{B}^{m+1}} = f$$

be of class $C^{k,1}$, $k \geq m-n$. Then $\text{rank } dF = n+1$ on an open set.

Here, by $C^{k,1}$ we denote the class of C^k functions whose k th order derivatives are Lipschitz continuous.

Indeed, if $\text{rank } dF \leq n$ everywhere, then all points in \mathbb{B}^{m+1} are critical and according to the Sard theorem [2], [5, Theorem 1.4], the measure of the set

$$\mathbb{B}^{n+1} \subset F(\mathbb{B}^{m+1}) = F(\text{Crit } F)$$

equals zero which is a clear contradiction.

In view of the above discussion it would be natural to expect that if $0 \neq [f] \in \pi_m(\mathbb{S}^n)$ then any Lipschitz extension F should satisfy $\text{rank } dF = n+1$ on a set of positive measure. However, the result of Wenger and Young [28, Theorem 2] shows that this is not always the case, see Corollary 1.6. Their proof employs an argument of Kaufman [22], who constructed a surprising example of a surjective mapping $F \in C^1(\mathbb{R}^{n+1}, \mathbb{R}^n)$ with $\text{rank } dF \leq 1$ everywhere.

One motivation for studying Lipschitz homotopy groups stems from the problem of approximation of Sobolev mappings. In the classical setting the answer to the question whether smooth or equivalently Lipschitz mappings $\text{Lip}(\mathcal{M}, \mathcal{N})$ between compact Riemannian manifolds

are dense in the Sobolev space of mappings $W^{1,p}(\mathcal{M}, \mathcal{N})$ heavily depends on the homotopy groups of \mathcal{N} , see [3, 4, 13, 14, 16, 19]. Here \mathcal{M} may have boundary, but $\partial\mathcal{N} = \emptyset$. More precisely, if $1 \leq p < \dim \mathcal{M}$ and $\pi_{[p]}(\mathcal{N}) \neq \{0\}$, where $[p]$ is the integral part of p , then Lipschitz maps are not dense in $W^{1,p}(\mathcal{M}, \mathcal{N})$. In the case of Sobolev mappings into the Heisenberg group it appears that the density of Lipschitz mappings $\text{Lip}(\mathcal{M}, \mathbb{H}_n)$ in $W^{1,p}(\mathcal{M}, \mathbb{H}_n)$, where \mathcal{M} is a compact Riemannian manifold with or without boundary, depends on Lipschitz homotopy groups of \mathbb{H}_n . For example it was proven in [8] that if $\dim \mathcal{M} \geq n + 1$ and $n \leq p < n + 1$, then Lipschitz maps $\text{Lip}(\mathcal{M}, \mathbb{H}_n)$ are not dense in $W^{1,p}(\mathcal{M}, \mathbb{H}_n)$. Note that in this case $\pi_{[p]}^{\text{Lip}}(\mathbb{H}_n) = \pi_n^{\text{Lip}}(\mathbb{H}_n) \neq \{0\}$. In this paper we extend this result as follows.

Theorem 1.9. *If \mathcal{M} is a compact Riemannian manifold with or without boundary of dimension $\dim \mathcal{M} \geq 4n$, then Lipschitz mappings $\text{Lip}(\mathcal{M}, \mathbb{H}_{2n})$ are not dense in $W^{1,p}(\mathcal{M}, \mathbb{H}_{2n})$ when $4n - 1 \leq p < 4n$.*

Again, according to Theorem 1.7, $\pi_{[p]}^{\text{Lip}}(\mathbb{H}_{2n}) = \pi_{4n-1}^{\text{Lip}}(\mathbb{H}_{2n}) \neq \{0\}$. On the other hand we would like to point out that it is possible to construct a smooth manifold \mathcal{N} with one point singularity such that all its Lipschitz homotopy groups are trivial, yet Lipschitz mappings into \mathcal{N} are not dense in the space of Sobolev mappings into \mathcal{N} , see [18].

The paper is organized as follows. In Section 2 we provide a brief introduction to the Heisenberg group and we prove Proposition 1.1. This proof is well known, but we recall it here for the sake of completeness and to see how the language of differential forms and their weak exterior derivatives can be used. Such an approach is an essential ingredient in this paper. In Section 3 we briefly recall the definition of Sobolev mappings into \mathbb{H}_n . In Section 4 we collect basic results about differential forms, DeRham cohomology and Sobolev spaces. We use these facts to generalize in Section 5 the Hopf invariant to Lipschitz mappings into Euclidean spaces whose derivative has low rank. That is essential for the proof of Theorem 1.7 which is done in Section 6. Finally in Section 7 we prove Theorem 1.9.

Those who are interested in the generalized Hopf invariant and its applications to homotopy groups of spheres may skip Sections 2 and 3 and read Sections 4-6. This material is of independent interest and it does not involve Heisenberg groups.

2. THE HEISENBERG GROUP

The *Heisenberg group* is a Lie group $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R} = \mathbb{R}^{2n+1}$ equipped with the group law

$$(z, t) * (z', t') = \left(z + z', t + t' + 2 \operatorname{Im} \left(\sum_{j=1}^n z_j \overline{z'_j} \right) \right).$$

A basis of left invariant vector fields is given by

$$(2.1) \quad X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \dots, n, \quad \text{and} \quad T = \frac{\partial}{\partial t}.$$

Here and in what follows we use notation

$$(z, t) = (z_1, \dots, z_n, t) = (x_1, y_1, \dots, x_n, y_n, t).$$

The Heisenberg group is equipped with the *horizontal distribution* $H\mathbb{H}_n$, which is defined at every point $p \in \mathbb{H}_n$ by

$$H_p\mathbb{H}_n = \operatorname{span} \{X_1(p), \dots, X_n(p), Y_1(p), \dots, Y_n(p)\}.$$

The distribution $H\mathbb{H}_n$ is equipped with the left invariant metric \mathbf{g} such that the vectors $X_1(p), \dots, X_n(p), Y_1(p), \dots, Y_n(p)$ are orthonormal at every point $p \in \mathbb{H}_n$. An absolutely continuous curve $\gamma : [a, b] \rightarrow \mathbb{H}_n$ is called *horizontal* if $\gamma'(s) \in H_{\gamma(s)}\mathbb{H}_n$ for almost every s . The Heisenberg group \mathbb{H}_n is equipped with the *Carnot-Carathéodory metric* d_{cc} which is defined as the infimum of the lengths of horizontal curves connecting two given points. The length of the curve is computed with respect to the metric \mathbf{g} on $H\mathbb{H}_n$. It is well known that any two points in \mathbb{H}_n can be connected by a horizontal curve and hence d_{cc} is a true metric. Actually, d_{cc} is topologically equivalent to the Euclidean metric. Moreover, for any compact set K there is a constant $C \geq 1$ such that

$$(2.2) \quad C^{-1}|p - q| \leq d_{cc}(p, q) \leq C|p - q|^{1/2}$$

for all $p, q \in K$. In what follows \mathbb{H}_n will always be regarded as the metric space (\mathbb{H}_n, d_{cc}) . It follows from (2.2) that the identity mapping from \mathbb{H}_n to \mathbb{R}^{2n+1} is locally Lipschitz, but its inverse is only locally Hölder continuous with exponent $1/2$. The Hausdorff dimension of any open set in \mathbb{H}_n equals $2n + 2$ and hence \mathbb{H}_n is not bi-Lipschitz homeomorphic to \mathbb{R}^{2n+1} , not even locally.

Proof of Proposition 1.1. If $f = (f_1, g_1, \dots, f_n, g_n, h) : \mathbb{R}^k \supset \Omega \rightarrow \mathbb{H}_n$ is a Lipschitz mapping, then it is locally Lipschitz as a mapping into \mathbb{R}^{2n+1} and hence it is differentiable a.e. It follows that the derivative of f is horizontal, i.e. $df(p)$ maps the tangent space $T_p\mathbb{R}^k$ into the

horizontal space $H_{f(p)}\mathbb{H}_n \subset T_{f(p)}\mathbb{R}^{2n+1}$. Indeed, f maps straight lines into Lipschitz curves and the Lipschitz curves in \mathbb{H}_n are horizontal, [17, Proposition 11.4]. Thus df maps vectors tangent to straight lines into vectors tangent to horizontal curves. Hence $df(p)(T_p\mathbb{R}^k) \subset H_{f(p)}\mathbb{H}_n$ for a.e. $p \in \mathbb{R}^k$. Let

$$(2.3) \quad \alpha = dt + 2 \sum_j (x_j dy_j - y_j dx_j)$$

be the standard contact form on \mathbb{R}^{2n+1} . It is easy to see that the kernel of $\alpha(p)$, $p \in \mathbb{R}^{2n+1}$, i.e. the collection of all vectors v such that $\alpha(p)v = 0$ coincides with the horizontal space $H_p\mathbb{H}_n$. Hence horizontality of the derivative of f means that $f^*\alpha(p) = 0$, for a.e. p , i.e.

$$(2.4) \quad dh + 2 \sum_{j=1}^n (f_j dg_j - g_j df_j) = 0 \quad \text{a.e.}$$

Since the functions are Lipschitz continuous we can take the distributional exterior derivative (see Lemma 4.1) and we get

$$\sum_{j=1}^n df_j \wedge dg_j = 0.$$

In other words if $\omega = \sum_j dx_j \wedge dy_j$ is a symplectic form on \mathbb{R}^{2n} and $F = (f_1, g_1, \dots, f_n, g_n)$ is a composition of f with the projection onto \mathbb{R}^{2n} , then $F^*\omega = 0$ a.e. as a pointwise equality. Let $\mathcal{J} : T_q\mathbb{R}^{2n} \rightarrow T_q\mathbb{R}^{2n}$ be given by

$$\mathcal{J} \left(\sum_{j=1}^n \left(a_j \frac{\partial}{\partial x_j} + b_j \frac{\partial}{\partial y_j} \right) \right) = \sum_{j=1}^n \left(-b_j \frac{\partial}{\partial x_j} + a_j \frac{\partial}{\partial y_j} \right).$$

Then for any vectors $u, v \in T_q\mathbb{R}^{2n}$ we have $\omega(q)(u, v) = -\langle u, \mathcal{J}v \rangle$. If f is differentiable at a point $p \in \mathbb{R}^k$ and $(F^*\omega)(p) = 0$, then for any vectors $u, v \in V := dF(p)(T_p\mathbb{R}^k) \subset T_{F(p)}\mathbb{R}^{2n}$ we have

$$\omega(F(p))(u, v) = -\langle u, \mathcal{J}v \rangle = 0.$$

Thus the space V is orthogonal to $\mathcal{J}V$ and hence $\dim V \leq n$. The rows of the matrix df are $\nabla f_1, \nabla g_1, \dots, \nabla f_n, \nabla g_n, \nabla h$. We proved that the rank of the minor formed by the first $2n$ rows is at most n . According to (2.4) the last row is linearly dependent on the first $2n$ rows and hence $\text{rank } df \leq n$ a.e. The proof is complete. \square

3. SOBOLEV MAPPINGS INTO \mathbb{H}_n

In this section we briefly recall the definition of the space of Sobolev mappings into \mathbb{H}_n . For more details, see [8]. If $\Omega \subset \mathbb{R}^m$ is open and V is a Banach space, then the space of vector valued Sobolev functions $W^{1,p}(\Omega, V)$ can be defined with the notion of Bochner integral and weak derivatives. $W^{1,p}(\Omega, V)$ is a Banach space. Using local coordinate systems one can easily extend this definition to the case of mappings from a compact manifold $W^{1,p}(\mathcal{M}, V)$. Any separable metric space and in particular the Heisenberg group \mathbb{H}_n admits an isometric embedding into ℓ^∞ (the Kuratowski embedding). Thus we can assume that $\mathbb{H}_n \subset \ell^\infty$. Then we define

$$W^{1,p}(\mathcal{M}, \mathbb{H}_n) = \{u \in W^{1,p}(\mathcal{M}, \ell^\infty) : u(x) \in \mathbb{H}_n \text{ a.e.}\}.$$

The space $W^{1,p}(\mathcal{M}, \mathbb{H}_n)$ is equipped with the norm metric $\rho(u, v) = \|u - v\|_{W^{1,p}}$. The question is whether Lipschitz mappings $\text{Lip}(\mathcal{M}, \mathbb{H}_n)$ form a dense subset of $W^{1,p}(\mathcal{M}, \mathbb{H}_n)$, see Theorem 1.9.

The following characterization of bounded Sobolev mappings into \mathbb{H}_n was proved in [7], [8, Proposition 6.8].

Proposition 3.1. *A bounded function*

$$f = (z, t) = (x_1, y_1, \dots, x_n, y_n, t) : \Omega \rightarrow \mathbb{H}_n$$

lies in $W^{1,p}(\Omega, \mathbb{H}_n)$ if and only if f is an element of the usual Sobolev space $W^{1,p}(\Omega, \mathbb{R}^{2n+1})$ and satisfies the contact equation

$$(3.1) \quad \nabla t = 2 \sum_{j=1}^n (y_j \nabla x_j - x_j \nabla y_j)$$

a.e. in Ω .

Thus the derivative of a Sobolev mapping $f = f(u_1, \dots, u_m)$ maps the tangent space to a horizontal subspace of \mathbb{H}_n . The length of the gradient ∇f can be computed with respect to the Euclidean metric $|\nabla f|$ in \mathbb{R}^{2n+1} or with respect to the sub-Riemannian metric in \mathbb{H}_n

$$|\nabla f|_{\mathbb{H}} = \left(\sum_{k=1}^m \left| \frac{\partial f}{\partial u_k} \right|_{\mathbb{H}}^2 \right)^{1/2},$$

where $|v|_{\mathbb{H}}$ stands for the length of the horizontal vector with respect to the given metric in the horizontal distribution. If the image of the mapping f is contained in a bounded subset of \mathbb{H}_n , then both lengths

$|\nabla f|$ and $|\nabla f|_{\mathbb{H}}$ are comparable. The following result was proved in [8, Theorem 1.6].

Proposition 3.2. *Let Ω be a bounded domain in \mathbb{R}^m . Suppose that $f_k, f \in W^{1,p}(\Omega, \mathbb{H}_n)$, $k = 1, 2, \dots$, $1 \leq p < \infty$, $f_k \rightarrow f$ in $W^{1,p}(\Omega, \mathbb{H}_n)$. Then*

$$\int_{\{f_k - f \notin Z\}} |\nabla f_k|_{\mathbb{H}}^p + |\nabla f|_{\mathbb{H}}^p \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

where Z denotes the center of \mathbb{H}_n .

Recall that the center of \mathbb{H}_n is the t -axis

$$(3.2) \quad Z = \{(z, t) \in \mathbb{H}_n : z = 0\}.$$

This result implies that on large sets the difference $f_k - f$ must belong to Z . That is a surprisingly strong condition and it stems from the fact that the Kuratowski embedding of \mathbb{H}_n into ℓ^∞ is highly non-smooth. The identity map $\text{id} : \mathbb{H}_n \rightarrow \mathbb{R}^{2n+1}$ is locally Lipschitz and hence if we assume in addition that mappings f_k, f are bounded, then $f, f_k \in W^{1,p}(\Omega, \mathbb{R}^{2n+1})$. However, it is not obvious that the convergence $f_k \rightarrow f$ in $W^{1,p}(\Omega, \mathbb{H}_n)$ implies convergence in $W^{1,p}(\Omega, \mathbb{R}^{2n+1})$, because in general the composition with a Lipschitz function need not be continuous in the Sobolev norm [15, Theorem 1.2]. However the following result is a consequence of Proposition 3.2, see [8, Corollary 1.7].

Corollary 3.3. *Let \mathcal{M} be a compact Riemannian manifold. Suppose that $f_k, f \in W^{1,p}(\mathcal{M}, \mathbb{H}_n)$, $k = 1, 2, \dots$, are uniformly bounded (i.e. the range of all the mappings is contained in a bounded subset of \mathbb{H}_n). If $f_k \rightarrow f$ in $W^{1,p}(\mathcal{M}, \mathbb{H}_n)$, then $f_k \rightarrow f$ in $W^{1,p}(\mathcal{M}, \mathbb{R}^{2n+1})$.*

We will also need the following fact [8, Lemma 6.5].

Lemma 3.4. *Let $f, g \in W^{1,p}(\Omega, \mathbb{H}_n)$. Let S be the set of points $p \in \Omega$ for which $f(p) - g(p) \in Z$. Then $\nabla f = \nabla g$ a.e. in S .*

4. DIFFERENTIAL FORMS, SOBOLEV SPACES, AND DERHAM COHOMOLOGY

In this section, we recall some notation and properties of differential forms on manifolds, with the goal of showing that if the DeRham cohomology is zero, then also the L^p -DeRham cohomology is zero, see Proposition 4.5, a result which essentially follows from the L^p -Hodge decomposition in [21, 24].

Before we start, we need to fix some notation. Let \mathcal{M} and \mathcal{N} be C^∞ -smooth oriented Riemannian manifolds with or without boundary. The volume form will be denoted by $d\text{vol}$. For smooth mappings $f : \mathcal{M} \rightarrow \mathcal{N}$ we let $f^* : C^\infty(\Lambda^\ell \mathcal{N}) \rightarrow C^\infty(\Lambda^\ell \mathcal{M})$ be the pull-back of ℓ -forms. By d we denote the derivative of smooth mappings, $d : C^\infty(\mathcal{M}, \mathcal{N}) \rightarrow C^\infty(T\mathcal{M}, T\mathcal{N})$, as well as the exterior derivative of ℓ -forms, $d : C^\infty(\Lambda^\ell \mathcal{M}) \rightarrow C^\infty(\Lambda^{\ell+1} \mathcal{M})$. The Hodge operator and the co-differential will be denoted by $*\omega$ and $\delta\omega$, respectively.

Any exterior ℓ -form $\omega \in \Lambda^\ell \mathcal{M}$ can be expressed in local coordinates $x = (x_1, \dots, x_k) : U \subset \mathcal{M} \rightarrow \mathbb{R}^k$ by

$$\omega = \sum_{1 \leq i_1 < i_2 < \dots < i_\ell \leq k} \omega_{i_1, i_2, \dots, i_\ell} dx_{i_1} \wedge \dots \wedge dx_{i_\ell} \quad \text{in } U,$$

and we say that $\omega \in C^\infty(\Lambda^\ell \mathcal{M})$, $\text{Lip}(\Lambda^\ell \mathcal{M})$, $L^p(\Lambda^\ell \mathcal{M})$, $W^{1,p}(\Lambda^\ell \mathcal{M})$ if the coefficients $\omega_{i_1, i_2, \dots, i_\ell}$ belong to the corresponding space. Here $W^{1,p}$ is the standard Sobolev space. The expression $C_0^\infty(\Lambda^\ell \mathcal{M})$ will stand for smooth ℓ -forms with compact support. In the case of manifolds with boundary we require the support to be disjoint from the boundary.

We will make frequent use of

$$(4.1) \quad f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta,$$

and

$$(4.2) \quad d(f^*\eta) = f^*(d\eta).$$

Note that (4.1), (4.2) also hold in a weak sense, in fact we have

Lemma 4.1. *Let \mathcal{M} be a smooth, k -dimensional oriented manifold with or without boundary.*

- (1) *If $f \in W_{\text{loc}}^{1,1}(\mathcal{M}, \mathbb{R}^m)$, then (4.1) holds pointwise almost everywhere.*
- (2) *If $f \in W_{\text{loc}}^{1,p}(\mathcal{M}, \mathbb{R}^m)$, $p \geq \ell + 1$, $0 \leq \ell \leq k - 1$, and $\eta \in C^\infty(\Lambda^\ell \mathbb{R}^m) \cap W^{1,\infty}$ (i.e. η and $|\nabla\eta|$ are bounded), then (4.2) holds in the weak sense, i.e.*

$$\int_{\mathcal{M}} f^*\eta \wedge d\varphi = (-1)^{\ell+1} \int_{\mathcal{M}} f^*(d\eta) \wedge \varphi$$

for all $\varphi \in C_0^\infty(\Lambda^{k-\ell-1} \mathcal{M})$.

- (3) If $\eta \in W_{\text{loc}}^{1,p}(\wedge^{\ell_1} \mathcal{M})$, $\omega \in W_{\text{loc}}^{1,p}(\wedge^{\ell_2} \mathcal{M})$, $\ell_1 + \ell_2 \leq k - 2$, $p \geq 2$, then $d(\eta \wedge d\omega) = d\eta \wedge d\omega$ weakly in the sense that

$$\int_{\mathcal{M}} \eta \wedge d\omega \wedge d\varphi = (-1)^{\ell_1 + \ell_2} \int_{\mathcal{M}} d\eta \wedge d\omega \wedge \varphi$$

for all $\varphi \in C_0^\infty(\wedge^{k-\ell_1-\ell_2-2} \mathcal{M})$.

Remark 4.2. In particular (1) and (2) hold under the condition that $f \in W_{\text{loc}}^{1,k}$ and (3) holds under the assumption that $\eta, \omega \in W_{\text{loc}}^{1,k}$. This is what we will need later on.

Proof. (1) is obvious. Regarding (2), observe that $d(f^*\eta)$ is not necessarily well defined in the pointwise sense since $f^*\eta$ is only in $L_{\text{loc}}^{p/\ell}$. Thus, we need to interpret the statement in the weak sense. Let f_ε be a smooth approximation of f in $W_{\text{loc}}^{1,p}$. Integration by parts gives

$$\int_{\mathcal{M}} f_\varepsilon^* \eta \wedge d\varphi = (-1)^{\ell+1} \int_{\mathcal{M}} f_\varepsilon^*(d\eta) \wedge \varphi$$

and the result follows by letting $\varepsilon \rightarrow 0$. The proof of (3) is similar. Let ω_ε and η_ε be smooth approximations of ω and η in $W_{\text{loc}}^{1,p}$. Integration by parts gives

$$\int_{\mathcal{M}} \eta_\varepsilon \wedge d\omega_\varepsilon \wedge d\varphi = (-1)^{\ell_1 + \ell_2} \int_{\mathcal{M}} d\eta_\varepsilon \wedge d\omega_\varepsilon \wedge \varphi$$

and the result follows by letting $\varepsilon \rightarrow 0$. \square

Also, we have the following version of the fundamental lemma of the calculus of variations.

Lemma 4.3. Assume \mathcal{M} to be a smooth, k -dimensional oriented manifold with or without boundary, and let $\eta \in L_{\text{loc}}^1(\wedge^\ell \mathcal{M})$ be such that

$$\int_{\mathcal{M}} \eta \wedge \varphi = 0 \quad \text{for all } \varphi \in C_0^\infty(\wedge^{k-\ell} \mathcal{M}).$$

Then $\eta = 0$ almost everywhere in \mathcal{M} .

Proof. Let $U \subset \mathcal{M}$ be a coordinate patch with coordinate functions $x = (x_1, \dots, x_k) : U \rightarrow \mathbb{R}^k$. Then

$$\eta = \sum_{1 \leq i_1 < \dots < i_\ell \leq k} f_{i_1 \dots i_\ell} dx_{i_1} \wedge \dots \wedge dx_{i_\ell} \quad \text{in } U.$$

It suffices to show that $f_{i_1 \dots i_\ell} \circ x^{-1} = 0$ a.e. in $x(U)$. Fix $1 \leq i_1 < \dots < i_\ell \leq k$. For a given $\psi \in C_0^\infty(x(U))$, let

$$\varphi := \psi \circ x \mid \det Dx \mid dx_{j_1} \wedge \dots \wedge dx_{j_{k-\ell}},$$

where $\{j_1, \dots, j_{k-\ell}\} = \{1, 2, \dots, k\} \setminus \{i_1, \dots, i_\ell\}$. Then $\varphi \in C_0^\infty(\wedge^{k-\ell} \mathcal{M})$ and consequently

$$0 = \int_{\mathcal{M}} \eta \wedge \varphi = \pm \int_{\mathcal{M}} f_{i_1 \dots i_\ell} \cdot \psi \circ x \cdot |\det Dx| \, d\text{vol} \equiv \pm \int_{x(U)} f_{i_1 \dots i_\ell} \circ x^{-1} \cdot \psi.$$

Since $f_{i_1 \dots i_\ell} \circ x^{-1} \in L_{\text{loc}}^1(x(U))$, and $\psi \in C_0^\infty(x(U))$ can be chosen arbitrarily, the classical fundamental lemma of the calculus of variations implies that $f_{i_1 \dots i_\ell} \circ x^{-1} = 0$ almost everywhere in $x(U)$. \square

We will need the following L^p -Hodge decomposition [24, Proposition 6.5].

Lemma 4.4 (*L^p -Hodge Decomposition*). *Let \mathcal{M} be a smooth, compact, k -dimensional oriented manifold without boundary and let $\Omega \subset \mathcal{M}$ be an open subset. Then for any $p \in (1, \infty)$ and any ℓ -form $\eta \in L^p(\wedge^\ell \Omega)$, $1 \leq \ell \leq k$ there exist $\omega_1 \in W^{1,p}(\wedge^{\ell-1} \Omega)$, $\omega_2 \in W^{1,p}(\wedge^{\ell+1} \Omega)$ such that*

$$(4.3) \quad \eta = d\omega_1 + \delta\omega_2 + h$$

where $h \in C^\infty(\wedge^\ell \Omega)$ is closed $dh = 0$ and co-closed $\delta h = 0$ and hence harmonic.

Indeed, in the case when $\Omega = \mathcal{M}$ the result was proved in [24, Proposition 6.5] and in the case of a general open set we simply extend η to $L^p(\wedge^\ell \mathcal{M})$ by zero, apply the Hodge decomposition on \mathcal{M} and restrict all the resulting forms to Ω .

Note that the above result applies to the manifold $\mathcal{M} \times (0, 1)$ since it can be isometrically embedded into $\mathcal{M} \times \mathbb{S}^1$ as an open set. We will need this special case when we show (Proposition 5.8) that the Hopf invariant is in fact invariant under Lipschitz homotopies.

As an application of the Hodge decomposition we prove that if the DeRham cohomology of an open set $\Omega \subset \mathcal{M}$ is zero, then also the L^p -DeRham cohomology is zero. More precisely we will show

Proposition 4.5. *Let \mathcal{M} and Ω be as in Lemma 4.4. Suppose that $H_{DR}^\ell(\Omega) = \{0\}$ for some $1 \leq \ell \leq k$, i.e. every smooth closed ℓ -form on Ω is exact. Let $\eta \in L^p(\wedge^\ell \Omega)$, $p \in (1, \infty)$ be weakly closed, i.e.*

$$(4.4) \quad \int_{\mathcal{M}} \eta \wedge d\varphi = 0 \quad \text{for all } \varphi \in C_0^\infty(\wedge^{k-\ell-1} \Omega).$$

Then there exists $\omega \in W_{\text{loc}}^{1,p}(\wedge^{\ell-1} \Omega)$ such that

$$\eta = d\omega \quad \text{a.e.,}$$

i.e. η is exact in the weak sense.

If $\Omega = \mathcal{M}$ is compact without boundary, then $\omega \in W^{1,p}(\wedge^{\ell-1} \mathcal{M})$ with the estimate

$$(4.5) \quad \|\omega\|_{W^{1,p}(\mathcal{M})} \leq C \|\eta\|_{L^p(\mathcal{M})}.$$

Proof. From Lemma 4.4, we obtain $\omega_1 \in W^{1,p}(\wedge^{\ell-1} \Omega)$, $\omega_2 \in W^{1,p}(\wedge^{\ell+1} \Omega)$, $h \in C^\infty(\wedge^\ell \Omega)$, $dh = 0$, $\delta h = 0$, such that

$$\eta = d\omega_1 + \delta\omega_2 + h.$$

Since $h \in C^\infty(\wedge^\ell \Omega)$, $dh = 0$, and $H_{DR}^\ell(\Omega) = \{0\}$, there exists $\omega_3 \in C^\infty(\wedge^{\ell-1} \Omega)$ such that $d\omega_3 = h$. Consequently,

$$(4.6) \quad \eta = d(\omega_3 + \omega_1) + \delta\omega_2.$$

Note that for any $\varphi \in C_0^\infty(\wedge^{k-\ell-1} \Omega)$, and for any $f \in W_{loc}^{1,p}(\wedge^{\ell-1} \Omega)$, by approximation

$$(4.7) \quad \int_{\Omega} df \wedge d\varphi = \int_{\Omega} d(f \wedge d\varphi) = 0.$$

Hence, from (4.4) and (4.6) we infer that for any $\varphi \in C_0^\infty(\wedge^{k-\ell-1} \Omega)$,

$$\int_{\Omega} \delta\omega_2 \wedge d\varphi = \int_{\Omega} d(\omega_3 + \omega_1) \wedge d\varphi + \int_{\Omega} \delta\omega_2 \wedge d\varphi = \int_{\Omega} \eta \wedge d\varphi = 0,$$

i.e. $\delta\omega_2$ is weakly closed. In particular, for any $\varphi \in C_0^\infty(\wedge^{k-\ell} \Omega)$

$$\begin{aligned} \int_{\Omega} \delta\omega_2 \wedge (d\delta + \delta d)\varphi &= \int_{\Omega} \delta\omega_2 \wedge \delta d\varphi \\ &= \pm \int_{\Omega} *d * \omega_2 \wedge *d * d\varphi \\ &= \pm \int_{\Omega} d * \omega_2 \wedge d * d\varphi = 0, \end{aligned}$$

where the last equality again follows from approximation and integration by parts just like in (4.7). That is, in the weak sense

$$\Delta\delta\omega_2 = 0,$$

where Δ is the Laplace-Beltrami operator. Thus $\delta\omega_2$ is actually smooth, see, e.g., [26, Theorem 6.5] or (for the local version) §6.35 and Exercise 14 on p. 253 of [26]. Since $\delta\omega_2$ is weakly closed and smooth, it is closed in the usual sense $d(\delta\omega_2) = 0$. Again, $H_{DR}^\ell(\Omega) = 0$ implies that there is $\omega_4 \in C^\infty(\wedge^{\ell-1} \Omega)$ such that $d\omega_4 = \delta\omega_2$. We have shown that

$$\eta = d(\omega_3 + \omega_1 + \omega_4),$$

and

$$\tilde{\omega} := \omega_3 + \omega_1 + \omega_4 \in W^{1,p}(\bigwedge^{\ell-1} \Omega) + C^\infty(\bigwedge^{\ell-1} \Omega) \subset W_{\text{loc}}^{1,p}(\bigwedge^{\ell-1} \Omega).$$

If $\Omega \subset \mathcal{M}$ is any open subset, and we do not expect estimate (4.5), we choose $\omega := \tilde{\omega}$.

Note however, that this choice of ω is not unique. In fact, setting $\omega := \tilde{\omega} - \omega_5$ for any weakly closed $\omega_5 \in W^{1,p}$

$$d\omega = d\tilde{\omega} = \eta \quad \text{a.e. in } \Omega.$$

If $\Omega = \mathcal{M}$ is compact without boundary then $\tilde{\omega} \in W_{\text{loc}}^{1,p}(\bigwedge^{\ell-1} \mathcal{M}) = W^{1,p}(\bigwedge^{\ell-1} \mathcal{M})$. By [21, Theorem 6.4] there exist a weakly closed form ω_5 such that $\omega := \tilde{\omega} - \omega_5 \in W^{1,p}(\bigwedge^{\ell-1} \mathcal{M})$ satisfies

$$\|\omega\|_{W^{1,p}} = \|\tilde{\omega} - \omega_5\|_{W^{1,p}} \leq C \|d\tilde{\omega}\|_{L^p} = C \|\eta\|_{L^p}.$$

This concludes the proof of Proposition 4.5. \square

5. HOPF INVARIANT FOR LOW-RANK MAPPINGS

Let α be the volume form on \mathbb{S}^{2n} . Then for any smooth mapping $f : \mathbb{S}^{4n-1} \rightarrow \mathbb{S}^{2n}$ we have that $d(f^*\alpha) = f^*(d\alpha) = 0$, so $f^*\alpha = d\omega$ for some smooth $2n-1$ form ω , because $H_{DR}^{2n}(\mathbb{S}^{4n-1}) = \{0\}$. The classical Hopf invariant of f is defined via the Whitehead formula

$$(5.1) \quad \mathcal{H}f = \int_{\mathbb{S}^{4n-1}} \omega \wedge d\omega.$$

See [6] for details and basic properties.

Hopf [20, Satz II, Satz II'] proved the following important result.

Lemma 5.1. *For any $n \in \mathbb{N}$ there exists a smooth map $f : \mathbb{S}^{4n-1} \rightarrow \mathbb{S}^{2n}$, such that $\mathcal{H}f \neq 0$.*

In this section we will generalize the Hopf invariant to Lipschitz mappings $f : \mathbb{S}^{4n-1} \rightarrow \mathbb{R}^m$, $m \geq 2n+1$, with $\text{rank } df \leq 2n$ almost everywhere. Let us first give the construction for smooth f , $\text{rank } df \leq 2n$. Let α be any smooth $2n$ -form in \mathbb{R}^m . Since $\text{rank } df \leq 2n$ and $d\alpha$ is a $(2n+1)$ -form, we have

$$(5.2) \quad d(f^*\alpha) = f^*(d\alpha) = 0,$$

because the determinant of every $(2n+1)$ -dimensional minor of df has to be zero. Thus there exists a $(2n-1)$ -form ω , such that

$$(5.3) \quad d\omega = f^*\alpha.$$

The Hopf invariant of f is defined by

$$(5.4) \quad \mathcal{H}_\alpha f := \int_{\mathbb{S}^{4n-1}} \omega \wedge d\omega.$$

It depends on α , but we will show that $\mathcal{H}_\alpha f$ is independent of the particular choice of ω , and that it is actually invariant under Lipschitz homotopies with rank of the derivative less than or equal $2n$. Obviously, if f is a constant map, then $\mathcal{H}_\alpha f = 0$. Moreover,

Proposition 5.2. *Let \mathbb{S}^{2n} be isometrically embedded into \mathbb{R}^m , $m \geq 2n + 1$ and let α be the volume form of \mathbb{S}^{2n} smoothly extended to \mathbb{R}^m , then*

$$\mathcal{H}_\alpha f = \mathcal{H}f \quad \text{for any smooth } f : \mathbb{S}^{4n-1} \rightarrow \mathbb{S}^{2n} \subset \mathbb{R}^m,$$

where $\mathcal{H}f$ is the classical Hopf invariant defined in (5.1). In particular there is a smooth map $f : \mathbb{S}^{4n-1} \rightarrow \mathbb{R}^m$ such that $\mathcal{H}_\alpha f \neq 0$.

This is obvious, since $\text{rank } df \leq 2n$ and $f^*(\alpha|_{\mathbb{S}^{2n}}) = f^*(\alpha)$. The last statement follows from Lemma 5.1.

Remark 5.3. Observe that the Hopf invariant $\mathcal{H}_\alpha f$ is defined for mappings $f : \mathbb{S}^{4n-1} \rightarrow \mathbb{R}^m$. If we denote by $\mathbb{S}^{4n-1}(r) = r\mathbb{S}^{4n-1}$ the sphere of radius r centered at the origin, then for mappings $f : \mathbb{S}^{4n-1}(r) \rightarrow \mathbb{R}^m$ we set

$$\mathcal{H}_\alpha \left(f|_{\mathbb{S}^{4n-1}(r)} \right) := \mathcal{H}_\alpha(\tilde{f}_r),$$

where $\tilde{f}_r : \mathbb{S}^{4n-1} \rightarrow \mathbb{R}^m$ is defined by $\tilde{f}_r(x) = f(rx)$.

5.1. Construction for Lipschitz functions. In order to make our argument precise, we have to ensure that every step above makes sense also for non-smooth Lipschitz mappings. For instance, observe that $f^*\alpha$ is only bounded, so one has to interpret $d(f^*\alpha)$ in the weak sense.

This is a non-trivial technicality, as one cannot just approximate f by smooth functions without losing the rank condition, which is essential for the construction of ω .

First, we confirm that (5.2) holds in a weak sense.

Lemma 5.4. *Let \mathcal{M} be a smooth k -dimensional oriented manifold with or without boundary, and assume that $f : \mathcal{M} \rightarrow \mathbb{R}^m$ is a Lipschitz map with $\text{rank } df \leq 2n$ almost everywhere. Then for any smooth $2n$ -form $\eta \in C^\infty(\Lambda^{2n} \mathbb{R}^m)$, $f^*\eta$ is weakly closed, i.e.*

$$\int_{\mathcal{M}} (f^*\eta) \wedge d\varphi = 0 \quad \text{for any } \varphi \in C_0^\infty(\Lambda^{k-2n-1} \mathcal{M}).$$

Proof. Let $\varphi \in C_0^\infty(\bigwedge^{k-2n-1} \mathcal{M})$. Since f is Lipschitz, it is in particular in $W_{\text{loc}}^{1,k}(\mathcal{M}, \mathbb{R}^m)$, so by Lemma 4.1 we have

$$\int_{\mathcal{M}} (f^* \eta) \wedge d\varphi = - \int_{\mathcal{M}} f^*(d\eta) \wedge \varphi = 0.$$

The last equality follows from the fact that $\text{rank } df \leq 2n$ a.e. and hence $f^*(d\eta) = 0$ a.e. \square

Let α be any smooth $2n$ -form on \mathbb{R}^m , $m \geq 2n+1$ and let $f : \mathbb{S}^{4n-1} \rightarrow \mathbb{R}^m$ be Lipschitz with $\text{rank } df \leq 2n$ a.e. According to Lemma 5.4, $f^* \alpha$ is weakly closed. Since $f^* \alpha \in L^2(\bigwedge^{2n} \mathbb{S}^{4n-1})$, Proposition 4.5 and the fact that $H_{DR}^{2n}(\mathbb{S}^{4n-1}) = \{0\}$ imply that there is $\omega \in W^{1,2}(\bigwedge^{2n-1} \mathbb{S}^{4n-1})$ such that $d\omega = f^* \alpha$. Thus, the definition (5.4) makes sense also for Lipschitz continuous f . Moreover,

Proposition 5.5. *Let $\omega_1, \omega_2 \in W^{1,p}(\bigwedge^{2n-1} \mathbb{S}^{4n-1})$, for some $p \geq 2 - \frac{1}{2n}$, and assume that $d\omega_1 = d\omega_2$ almost everywhere. Then the forms $\omega_i \wedge d\omega_i$, $i = 1, 2$ are integrable and*

$$\int_{\mathbb{S}^{4n-1}} \omega_1 \wedge d\omega_1 = \int_{\mathbb{S}^{4n-1}} \omega_2 \wedge d\omega_2.$$

In particular, for any Lipschitz map $f : \mathbb{S}^{4n-1} \rightarrow \mathbb{R}^m$ with $\text{rank } df \leq 2n$ a.e., the definition (5.4) of $\mathcal{H}_\alpha f$ is independent of the choice of $\omega \in W^{1,p}(\bigwedge^{2n-1} \mathbb{S}^{4n-1})$ with $d\omega = f^ \alpha$.*

In the proof we will need the following lemma and its corollary.

Lemma 5.6. *If $f, g \in W^{1, \frac{2k}{k+1}}(\mathbb{R}^k)$, then $fg \in W^{1,1}(\mathbb{R}^k)$.*

Proof. Let $p = 2k/(k+1)$. If $k \geq 2$, easy calculation shows that the Sobolev exponent satisfies $p^* = p/(p-1)$ and hence $fg \in L^1$. This is also true for $k = 1$ since $W^{1,1}(\mathbb{R}) \subset L^\infty$. Sobolev functions are absolutely continuous on lines [10, Section 4.9]. Since the product of absolutely continuous functions is absolutely continuous, fg is also absolutely continuous on lines. Hence we can compute partial derivatives

$$\frac{\partial}{\partial x_i}(fg) = \frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i}.$$

Again, since $p^* = p/(p-1)$ we conclude that $\partial(fg)/\partial x_i \in L^1$ for $i = 1, 2, \dots, k$. The characterization of $W^{1,1}$ by absolute continuity on lines [10, Section 4.9] implies that $fg \in W^{1,1}(\mathbb{R}^k)$. \square

Corollary 5.7. *If $\omega_1, \omega_2 \in W^{1,p}(\bigwedge^{2n-1} \mathbb{S}^{4n-1})$, $p \geq 2 - \frac{1}{2n}$, then $\omega_1 \wedge \omega_2 \in W^{1,1}(\bigwedge^{4n-2} \mathbb{S}^{4n-1})$.*

Proof. If $k = 4n - 1$, then

$$\frac{2k}{k+1} = 2 - \frac{1}{2n}$$

and the result follows from the lemma applied to representations of ω_1 and ω_2 in local coordinates. \square

Proof of Proposition 5.5. The argument is a modification of the proof of [6, Proposition 17.22.(a)]. We can assume that $p = 2 - \frac{1}{2n}$. Let $\omega_1, \omega_2 \in W^{1,p}(\bigwedge^{2n-1} \mathbb{S}^{4n-1})$ be such that

$$d(\omega_1 - \omega_2) = 0 \quad \text{a.e.}$$

Since $\omega_1, \omega_2 \in W^{1,p}(\bigwedge^{2n-1} \mathbb{S}^{4n-1})$ with $p = 2 - \frac{1}{2n}$, by Sobolev embedding, $\omega_1, \omega_2 \in L^{\frac{p}{p-1}}(\bigwedge^{2n-1} \mathbb{S}^{4n-1})$, that is

$$\omega_i \wedge d\omega_j \in L^1(\bigwedge^{4n-1} \mathbb{S}^{4n-1}) \quad \text{for any } i, j \in \{1, 2\}.$$

Moreover we have

$d((\omega_1 - \omega_2) \wedge \omega_1) = d(\omega_1 - \omega_2) \wedge \omega_1 - (\omega_1 - \omega_2) \wedge d\omega_1 = -(\omega_1 - \omega_2) \wedge d\omega_1$ almost everywhere. Hence,

$$\begin{aligned} \int_{\mathbb{S}^{4n-1}} \omega_1 \wedge d\omega_1 - \int_{\mathbb{S}^{4n-1}} \omega_2 \wedge d\omega_2 &= \int_{\mathbb{S}^{4n-1}} (\omega_1 - \omega_2) \wedge d\omega_1 \\ &= - \int_{\mathbb{S}^{4n-1}} d((\omega_1 - \omega_2) \wedge \omega_1) \\ &= 0. \end{aligned}$$

The last step is Stokes' theorem, which holds obviously by approximating $(\omega_1 - \omega_2) \wedge \omega_1 \in W^{1,1}(\bigwedge^{4n-2} \mathbb{S}^{4n-1})$ by smooth $(4n-2)$ -forms. \square

Next, we show that $\mathcal{H}_\alpha f$ is invariant under Lipschitz homotopies of rank at most $2n$.

Proposition 5.8. *Let $f, g : \mathbb{S}^{4n-1} \rightarrow \mathbb{R}^m$ be two Lipschitz maps of rank at most $2n$ and assume that there is a Lipschitz homotopy $H : [0, 1] \times \mathbb{S}^{4n-1} \rightarrow \mathbb{R}^m$, $H(0, \cdot) = f(\cdot)$, $H(1, \cdot) = g(\cdot)$ such that $\text{rank } dH \leq 2n$ a.e. Then*

$$\mathcal{H}_\alpha f = \mathcal{H}_\alpha g.$$

Proof. We adapt the argument from [6, Proposition 17.22.(c)]. However, since we are dealing with non-smooth mappings we have to be very careful. We may assume that $H : [0, 1] \times \mathbb{S}^{4n-1} \rightarrow \mathbb{R}^m$ is constant

in t for $0 \leq t \leq 1/4$ and $3/4 \leq t \leq 1$. If not, we take a Lipschitz function $s(t) : [0, 1] \rightarrow [0, 1]$ such that

$$s(t) := \begin{cases} 0 & 0 \leq t \leq \frac{1}{4}, \\ 1 & \frac{3}{4} \leq t \leq 1, \end{cases}$$

and consider $H(s(t), x)$ instead of $H(t, x)$, which is still Lipschitz, and also satisfies the rank condition. We have

$$H^* \alpha \in L^\infty(\bigwedge^{2n}((0, 1) \times \mathbb{S}^{4n-1})).$$

Since \mathbb{S}^{4n-1} is a deformation retract of $(0, 1) \times \mathbb{S}^{4n-1}$ we conclude that $H_{DR}^{2n}((0, 1) \times \mathbb{S}^{4n-1}) = H_{DR}^{2n}(\mathbb{S}^{4n-1}) = \{0\}$, [6, Corollary 4.1.2.2]. Now from the fact that $\text{rank } dH \leq 2n$ and from Lemma 5.4 we infer that $H^* \alpha$ is weakly closed. Since $(0, 1) \times \mathbb{S}^{4n-1}$ can be isometrically embedded into the compact manifold $\mathbb{S}^1 \times \mathbb{S}^{4n-1}$ as an open set by Proposition 4.5 there is $\omega \in W_{\text{loc}}^{1,2}(\bigwedge^{2n-1}(0, 1) \times \mathbb{S}^{4n-1})$ such that

$$d\omega = H^* \alpha \quad \text{a.e.}$$

Denote by

$$\iota_t : \mathbb{S}^{4n-1} \rightarrow \{t\} \times \mathbb{S}^{4n-1} \subset (0, 1) \times \mathbb{S}^{4n-1}.$$

the canonical embedding of the sphere by the identity. From the Rademacher and the Fubini theorems it follows that for almost every $t \in (0, 1)$, H is differentiable at almost all points of the sphere $\{t\} \times \mathbb{S}^{4n-1}$. Thus the chain rule implies that

$$(5.5) \quad (H \circ \iota_t)^* \alpha = \iota_t^* H^* \alpha \quad \text{a.e. in } \mathbb{S}^{4n-1}$$

Note also that $\omega_t := \iota_t^* \omega$ is defined a.e. on \mathbb{S}^{4n-1} for almost all $t \in (0, 1)$. Approximate ω by

$$\omega^\varepsilon \in C^\infty \left(\bigwedge^{2n-1}(0, 1) \times \mathbb{S}^{4n-1} \right) \quad \text{in} \quad W_{\text{loc}}^{1,2} \left(\bigwedge^{2n-1}(0, 1) \times \mathbb{S}^{4n-1} \right).$$

It follows from the Fubini theorem (cf. [14, p. 189]) that there is a sequence $\varepsilon_i \rightarrow 0$ such that

$$(5.6) \quad \omega_t^{\varepsilon_i} := \iota_t^* \omega^{\varepsilon_i} \rightarrow \iota_t^* \omega = \omega_t \quad \text{in} \quad W^{1,2} \left(\bigwedge^{2n-1} \mathbb{S}^{4n-1} \right)$$

and

$$\iota_t^* d\omega^{\varepsilon_i} \rightarrow \iota_t^* d\omega \quad \text{in} \quad L^2 \left(\bigwedge^{2n} \mathbb{S}^{4n-1} \right)$$

for almost all $t \in (0, 1)$. Since

$$\iota_t^* d\omega^{\varepsilon_i} = d\iota_t^* \omega^{\varepsilon_i} \rightarrow d\omega_t \quad \text{in} \quad L^2$$

we conclude that

$$(5.7) \quad d\omega_t = \iota_t^* d\omega \quad \text{for a.e. } t \in (0, 1).$$

Fix $t_0 \in (0, 1/4)$ and $t_1 \in (3/4, 1)$ such that (5.5), (5.6) and (5.7) are satisfied. We have

$$d\omega_{t_0} = \iota_{t_0}^* d\omega = \iota_{t_0}^* H^* \alpha = (H \circ \iota_{t_0})^* \alpha = f^* \alpha \quad \text{a.e. in } \mathbb{S}^{4n-1}.$$

Similarly

$$d\omega_{t_1} = g^* \alpha \quad \text{a.e. in } \mathbb{S}^{4n-1}.$$

Hence

$$\begin{aligned} \mathcal{H}_\alpha f - \mathcal{H}_\alpha g &= \int_{\mathbb{S}^{4n-1}} \omega_{t_0} \wedge d\omega_{t_0} - \int_{\mathbb{S}^{4n-1}} \omega_{t_1} \wedge d\omega_{t_1} \\ &= \lim_{i \rightarrow \infty} \left(\int_{\mathbb{S}^{4n-1}} \omega_{t_0}^{\varepsilon_i} \wedge d\omega_{t_0}^{\varepsilon_i} - \int_{\mathbb{S}^{4n-1}} \omega_{t_1}^{\varepsilon_i} \wedge d\omega_{t_1}^{\varepsilon_i} \right) \\ &= \lim_{i \rightarrow \infty} \int_{\partial((t_0, t_1) \times \mathbb{S}^{4n-1})} \omega^{\varepsilon_i} \wedge d\omega^{\varepsilon_i} \\ &= \lim_{i \rightarrow \infty} \int_{(t_0, t_1) \times \mathbb{S}^{4n-1}} d(\omega^{\varepsilon_i} \wedge d\omega^{\varepsilon_i}) \\ &= \int_{(t_0, t_1) \times \mathbb{S}^{4n-1}} d\omega \wedge d\omega \\ &= \int_{(t_0, t_1) \times \mathbb{S}^{4n-1}} H^* \alpha \wedge H^* \alpha \\ &= \int_{(t_0, t_1) \times \mathbb{S}^{4n-1}} H^*(\alpha \wedge \alpha) = 0. \end{aligned}$$

The last equality follows from the fact that $\text{rank } dH \leq 2n$ a.e. and $\alpha \wedge \alpha$ is a $4n$ -form so $H^*(\alpha \wedge \alpha) = 0$ a.e. \square

We will also need the following convergence result.

Proposition 5.9. *Let $g_k, g \in \text{Lip}(\mathbb{S}^{4n-1}, \mathbb{R}^m)$ be Lipschitz mappings with $\text{rank } dg_k, \text{rank } dg \leq 2n$ almost everywhere and such that for a given $\alpha \in C^\infty(\wedge^{2n} \mathbb{R}^m)$*

$$\lim_{k \rightarrow \infty} \|g_k^* \alpha - g^* \alpha\|_{L^p(\wedge^{2n} \mathbb{S}^{4n-1})} = 0,$$

for some $p \geq 2 - \frac{1}{2n}$. Then

$$\lim_{k \rightarrow \infty} \mathcal{H}_\alpha g_k = \mathcal{H}_\alpha g.$$

Proof. We can assume that $p = 2 - \frac{1}{2n}$. According to Lemma 5.4 the forms $g^* \alpha$ and $g_k^* \alpha$ are weakly closed. Hence from Proposition 4.5 there

exist $\omega, \omega_k \in W^{1,p}(\wedge^{2n-1} \mathbb{S}^{4n-1})$ with $d\omega = g^*\alpha$, $d\omega_k = g_k^*\alpha$, and such that

$$\|\omega\|_{L^{\frac{p}{p-1}}} \leq C\|\omega\|_{W^{1,p}} \leq C'\|g^*\alpha\|_{L^p},$$

and similarly

$$\|\omega_k\|_{L^{\frac{p}{p-1}}} \leq C'\|g_k^*\alpha\|_{L^p}.$$

We used here the Sobolev inequality and the fact that $p^* = \frac{p}{p-1}$. In view of Proposition 5.5,

$$\mathcal{H}_\alpha g = \int_{\mathbb{S}^{4n-1}} \omega \wedge d\omega, \quad \mathcal{H}_\alpha g_k = \int_{\mathbb{S}^{4n-1}} \omega_k \wedge d\omega_k.$$

Hence

$$\begin{aligned} \mathcal{H}_\alpha g_k - \mathcal{H}_\alpha g &= \int_{\mathbb{S}^{4n-1}} \omega_k \wedge d\omega_k - \omega \wedge d\omega \\ &= \int_{\mathbb{S}^{4n-1}} \omega_k \wedge (d\omega_k - d\omega) + (\omega_k - \omega) \wedge d\omega \\ &= \int_{\mathbb{S}^{4n-1}} \omega_k \wedge (d\omega_k - d\omega) + d(\omega_k - \omega) \wedge \omega \\ &\leq C \left(\|\omega_k\|_{L^{\frac{p}{p-1}}} + \|\omega\|_{L^{\frac{p}{p-1}}} \right) \|g^*\alpha - g_k^*\alpha\|_{L^p} \\ &\leq C (\|g_k^*\alpha\|_{L^p} + \|g^*\alpha\|_{L^p}) \|g^*\alpha - g_k^*\alpha\|_{L^p} \xrightarrow{k \rightarrow \infty} 0. \end{aligned}$$

□

6. PROOF OF THEOREM 1.7

The case of $\pi_n(\mathbb{S}^n)$ having already been proved in Section 1, it remains to show that the homotopy group $\pi_{4n-1}(\mathbb{S}^{2n})$ is rank-essential for $n \in \mathbb{N}$. Let $f : \mathbb{S}^{4n-1} \rightarrow \mathbb{S}^{2n} \subset \mathbb{R}^{2n+1}$ be the mapping, and α the $2n$ -form on \mathbb{R}^{2n+1} such that

$$(6.1) \quad \mathcal{H}_\alpha f \neq 0.$$

See Proposition 5.2. Assume by contradiction that $\pi_{4n-1}(\mathbb{S}^{2n})$ is not rank-essential. Hence there exists a Lipschitz extension $F : \mathbb{B}^{4n} \rightarrow \mathbb{R}^{2n+1}$ such that $\text{rank } dF \leq 2n$ almost everywhere in \mathbb{B}^{4n} . Define the homotopy

$$H(t, \theta) : [0, 1] \times \mathbb{S}^{4n-1} \rightarrow \mathbb{R}^{2n+1}$$

between $f = H(1, \cdot)$ and a constant map $g = H(0, \cdot)$ via

$$H(t, \theta) := F(t\theta).$$

This homotopy is clearly Lipschitz and with $\text{rank } dH \leq 2n$. Obviously, $\text{rank } df, \text{rank } dg \leq 2n$. Then, since the Hopf invariant $\mathcal{H}_\alpha f$ does not change under Lipschitz rank $2n$ -homotopies, see Proposition 5.8,

$$\mathcal{H}_\alpha f = \mathcal{H}_\alpha g = 0,$$

which contradicts (6.1). The proof is complete. \square

7. PROOF OF THEOREM 1.9

The proof is similar to that of Theorem 1.2(a) and Proposition 1.3 in [8].

Assume first that $\mathcal{M} = \mathbb{B}^{4n}$. Let $\phi : \mathbb{S}^{2n} \rightarrow \mathbb{H}_{2n}$ be a bi-Lipschitz map, which is a smooth embedding as a map from \mathbb{S}^{2n} to \mathbb{R}^{4n+1} , see Proposition 1.2. Let $f_0 \in C^\infty(\mathbb{S}^{4n-1}, \mathbb{S}^{2n})$ be the Hopf map from Lemma 5.1 such that

$$\mathcal{H}f_0 \neq 0.$$

It easily follows from Proposition 3.1 that

$$f(x) := \phi \circ f_0 \left(\frac{x}{|x|} \right) \in W^{1,p}(\mathbb{B}^{4n}, \mathbb{H}_{2n}), \quad \text{for all } 1 \leq p < 4n.$$

We will prove that f cannot be approximated in $W^{1,p}(\mathbb{B}^{4n}, \mathbb{H}_{2n})$ by Lipschitz mappings $\text{Lip}(\mathbb{B}^{4n}, \mathbb{H}_{2n})$ when $4n - 1 \leq p < 4n$. Suppose to the contrary that there is a sequence $g_k \in \text{Lip}(\mathbb{B}^{4n}, \mathbb{H}_{2n})$ such that

$$g_k \rightarrow f \quad \text{in } W^{1,p}(\mathbb{B}^{4n}, \mathbb{H}_{2n}).$$

Note that by Proposition 1.1 both $\text{rank } dg_k$ and $\text{rank } df$ do not exceed $2n$. Formally, f is not Lipschitz, but it is locally Lipschitz away from the singularity at the origin and hence Proposition 1.1 applies to f as well.

Choose $\alpha \in C_0^\infty(\bigwedge^{2n} \mathbb{R}^{4n+1})$ to be a smooth extension of the push-forward $\phi_* d\text{vol}_{\mathbb{S}^{2n}}$. Recalling our definition of the Hopf invariant of mappings whose domains are scaled spheres $\mathbb{S}^{4n-1}(r)$, see Remark 5.3,

$$(7.1) \quad \mathcal{H}_\alpha \left(f|_{\mathbb{S}^{4n-1}(r)} \right) = \mathcal{H}(f_0) \neq 0 \quad \text{for all } r \in (0, 1).$$

On the other hand, $g_k \in \text{Lip}(\mathbb{B}^{4n}, \mathbb{H}_{2n})$, and hence $g_k|_{\mathbb{S}^{4n-1}(r)}$ as a mapping to \mathbb{R}^{4n+1} is Lipschitz homotopic to a constant map with the homotopy satisfying the rank condition $\text{rank } dH \leq 2n$ a.e. (see Proposition 1.1). Thus Proposition 5.8 yields

$$(7.2) \quad \mathcal{H}_\alpha \left(g_k|_{\mathbb{S}^{4n-1}(r)} \right) = 0 \quad \text{for all } k \text{ and all } r \in (0, 1).$$

We are now going to show that (7.1) and (7.2) contradict each other.

Since the mappings g_k are not necessarily uniformly bounded we cannot claim that $g_k \rightarrow f$ in $W^{1,p}(\mathbb{B}^{4n}, \mathbb{R}^{4n+1})$, see Corollary 3.3. In particular we cannot claim that $\nabla g_k \rightarrow \nabla f$ in $L^p(\mathbb{B}^{4n})$. Nevertheless we can assume upon passing to a subsequence that $g_k \rightarrow f$ a.e. in \mathbb{B}^{4n} . We will construct sets E_k such that

$$(7.3) \quad \chi_{E_k} \nabla g_k \rightarrow \nabla f \quad \text{in } L^p(\mathbb{B}^{4n}).$$

Let $K = \text{supp } \alpha$, let

$$S_k = \{x \in \mathbb{B}^{4n} : g_k(x) - f(x) \in Z\},$$

where Z is the center of \mathbb{H}_{2n} defined in (3.2), and let

$$E_k = S_k \cup g_k^{-1}(K).$$

We claim that (7.3) is true. According to Lemma 3.4, $\nabla g_k = \nabla f$ a.e. in S_k and hence

$$\int_{S_k} |\nabla f - \nabla g_k|^p = 0.$$

Since the mappings f and $g_k|_{g_k^{-1}(K)}$ are uniformly bounded, the Euclidean lengths $|\nabla f|$ and $|\chi_{E_k} \nabla g_k|$ are comparable to the Heisenberg lengths $|\nabla f|_{\mathbb{H}}$ and $|\chi_{E_k} \nabla g_k|_{\mathbb{H}}$ respectively on the set $\mathbb{B}^{4n} \setminus S_k$. Thus Proposition 3.2 yields

$$\int_{\mathbb{B}^{4n} \setminus S_k} |\nabla f|^p + |\chi_{E_k} \nabla g_k|^p \leq C \int_{\mathbb{B}^{4n} \setminus S_k} |\nabla f|_{\mathbb{H}}^p + |\chi_{E_k} \nabla g_k|_{\mathbb{H}}^p \rightarrow 0.$$

Hence

$$\begin{aligned} & \int_{\mathbb{B}^{4n}} |\nabla f - \chi_{E_k} \nabla g_k|^p \\ & \leq C \left(\int_{S_k} |\nabla f - \nabla g_k|^p + \int_{\mathbb{B}^{4n} \setminus S_k} |\nabla f|^p + |\chi_{E_k} \nabla g_k|^p \right) \rightarrow 0. \end{aligned}$$

Now it follows from the Fubini theorem that – up to a subsequence which we again denote by g_k – for almost any $r \in (0, 1)$

$$\chi_{E_k} \nabla g_k|_{\mathbb{S}^{4n-1}(r)} \xrightarrow{k \rightarrow \infty} \nabla f|_{\mathbb{S}^{4n-1}(r)} \quad \text{in } L^p(\mathbb{S}^{4n-1}(r))$$

This and the almost everywhere convergence $g_k \rightarrow f$ implies that for almost all $r \in (0, 1)$

$$(7.4) \quad \chi_{E_k} \left(g_k|_{\mathbb{S}^{4n-1}(r)} \right)^* \alpha \rightarrow \left(f|_{\mathbb{S}^{4n-1}(r)} \right)^* \alpha \quad \text{in } L^{p/2n} \left(\bigwedge^{2n} \mathbb{S}^{4n-1}(r) \right).$$

On the other hand since $K = \text{supp } \alpha$, $g_k^* \alpha = 0$ a.e. in $\mathbb{B}^{4n} \setminus g_k^{-1}(K)$ and hence $g_k^* \alpha = 0$ a.e. in $\mathbb{B}^{4n} \setminus E_k$. Accordingly, for almost all $r \in (0, 1)$

$$\left(g_k|_{\mathbb{S}^{4n-1}(r)}\right)^* \alpha = \chi_{E_k} \left(g_k|_{\mathbb{S}^{4n-1}(r)}\right)^* \alpha$$

which together with (7.4) yields

$$\lim_{k \rightarrow \infty} \left\| \left(g_k|_{\mathbb{S}^{4n-1}(r)}\right)^* \alpha - \left(f|_{\mathbb{S}^{4n-1}(r)}\right)^* \alpha \right\|_{L^{p/2n}(\wedge^{2n} \mathbb{S}^{4n-1}(r))} = 0.$$

This, (7.1), and Proposition 5.9 imply for $p \geq 4n - 1$ and almost all $r \in (0, 1)$

$$\lim_{k \rightarrow \infty} \mathcal{H}_\alpha \left(g_k|_{\mathbb{S}^{4n-1}(r)}\right) = \mathcal{H}_\alpha \left(f|_{\mathbb{S}^{4n-1}(r)}\right) \neq 0,$$

which contradicts (7.2).

If \mathcal{M} is a general manifold of dimension $\dim \mathcal{M} \geq 4n$, then the result follows from the case \mathbb{B}^{4n} by a simple surgery as in the proof of Theorem 1.2 in [8]. We simply construct a mapping $f \in W^{1,p}(\mathcal{M}, \mathbb{H}_{2n})$ such that on a family of $4n$ dimensional slices in \mathcal{M} it coincides with the mapping constructed above. By using the Fubini theorem one easily arrives to a contradiction by employing the case of \mathbb{B}^{4n} . \square

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