Hard analysis meets critical knots
Stationary points of the Möbius energy are smooth

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We prove that if a curve $\gamma \in H^{3/2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ parametrized by arc length is a stationary point of the Möbius energy introduced by Jun O'Hara in [O'H91], then γ is smooth. Our methods only rely on purely analytical arguments, entirely without using Möbius invariance. Furthermore, they are not fundamentally restricted to one-dimensional domains, but are generalizable to arbitrary dimensions.

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1. Introduction

To find nice geometric representatives within a given knot class, several new energies have been invented in the last two decades. The earliest knot energy for smooth curves was the so-called *Möbius energy* introduced by Jun O'Hara in [O'H91],

$$E^{(2)}(\gamma) := \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left(\frac{1}{|\gamma(u+w)-\gamma(u)|^2} - \frac{1}{d_\gamma(u+w,u)^2} \right) \left|\gamma'(u+w)\right|\left|\gamma'(u)\right| \, \mathrm{d}w \, \mathrm{d}u$$

which was later on extended to the family of energies

$$E^{(\alpha,p)}(\gamma) := \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left(\frac{1}{|\gamma(u+w)-\gamma(u)|^\alpha} - \frac{1}{d_\gamma(u+w,u)^\alpha} \right)^p \left|\gamma'(u+w)\right|\left|\gamma'(u)\right| \, \mathrm{d}w \, \mathrm{d}u$$

for $\alpha, p \in [1,\infty)$, see [O'H94]. Here $d_\gamma(u+w,u)$ denotes the intrinsic distance between $\gamma(u+w)$ and $\gamma(u)$ on the curve γ. More precisely,

$$d_\gamma(u+w,u) := \min\left(\mathscr{L}(\gamma|_{[u,u+w]}), \mathscr{L}(\gamma) - \mathscr{L}(\gamma|_{[u,u+w]})\right) \tag{1.1}$$

provided $|w| \le \frac{1}{2}$ where $\mathscr{L}(\gamma) := \int_0^1 |\gamma'(\theta)| \, \mathrm{d}\theta$ is the length of γ.

Crucially using the Möbius invariance of this knot energy, Michael Freedman, Zheng-Xu He, and Zhenghan Wang [FHW94] were able to show that there are minimizers of the Möbius energy within every prime knot class and that these are in fact of class $C^{1,1}$. More precisely, they could show that if γ is a local minimizer with respect to the L^∞-topology, and if γ is parametrized by arc length, then γ is $C^{1,1}$. Together with a bootstrapping argument due to He [He00], one then obtains that local minimizers of the Möbius energy are smooth, also see [Rt10].

Unfortunately, motivated by numerical evidence, Rob Kusner and John Sullivan were led to conjecture that there are no minimizers within composite knot classes [KS97]. In contrast to that, there are minimizers of the energies $E^{(\alpha,p)}$ in the case that $jp > 2$ as shown in [O'H94].

In this article, we will prove that even only stationary points of the Möbius energy are of class C^∞ under the mildest condition one can think of: that $E^{(2)}(\gamma)$ is finite - an assumption which, as shown in a recent work of the first author [Bla12], is equivalent to assuming that γ is an injective curve of class $H^{\frac{3}{2}}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$. Our motivation to do so is twofold: First of all, of course, this is a much stronger result than the smoothness of local minimizers as stated above. Secondly, the Möbius invariance is not essential for proving smoothness of local minimizers as we do not need it in our arguments. Thus there is the chance to study other, possibly not Möbius invariant, critical knot energies, using the techniques developed in this article. Additionally, our arguments are not restricted to the one-dimensional situation but can be applied to arbitrary dimensions.

The price we pay is that, instead of the very appealing geometric argument in [FHW94], we have to adapt some sophisticated techniques originally developed by Tristan Rivière and Francesca Da Lio [DLR11a, DLR11b, DL11] and the third author [Sch12, Sch11] to deal with $\frac{n}{2}$- harmonic maps into manifolds.

The first task in order to prove this result, is to derive the Euler-Lagrange equation for such stationary points. In [FHW94], it was shown that for simple closed curves $\gamma \in C^{1,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ and $h \in C^{1,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ we have

$$\begin{aligned}\delta E^{(2)}(\gamma; h) &:= \lim_{\tau \searrow 0} \frac{E^{(2)}(\gamma + \tau h) - E^{(2)}(\gamma)}{\tau} \\ &= \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \left(\frac{\gamma'(u)h'(u) - \frac{\langle \gamma(u)-\gamma(v), h(u)-h(v)\rangle}{|\gamma(u)-\gamma(v)|^2}}{|\gamma(u) - \gamma(v)|^2} \right) |\gamma'(v)|\, |\gamma'(u)|\, dv\, du.\end{aligned}$$

We will show that this formula is still valid under the weaker assumption that $\gamma \in H^{3/2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ with $\gamma' \in L^\infty$, i.e., for arc-length parametrized γ we only assume that the Möbius energy is finite. We call a curve *regular* if there is a positive constant $c = c(\gamma)$ with $|\gamma'(x)| \geq c$ for all $x \in \mathbb{R}/\mathbb{Z}$.

Theorem I ($E^{(2)} \in C^1(H^{3/2}_{\text{ir}} \cap H^{1,\infty})$). *The energy $E^{(2)}$ is continuously differentiable on the space of injective and regular curves belonging to $H^{3/2} \cap H^{1,\infty}$. Furthermore, if $\gamma \in H^{3/2}$ is injective and parametrized by arc-length and $\varphi \in H^{3/2} \cap H^{1,\infty}$ the first variation*

$$\delta E^{(2)}(\gamma; \varphi) := \lim_{\tau \to 0} \frac{E^{(2)}(\gamma + \tau\varphi) - E^{(2)}(\gamma)}{\tau}$$

exists and equals

$$2 \lim_{\varepsilon \searrow 0} \iint_{U_\varepsilon} \left(\langle \gamma'(u), \varphi'(u) \rangle - \frac{\langle \gamma(u+w) - \gamma(u), \varphi(u+w) - \varphi(u) \rangle}{|\gamma(u+w) - \gamma(u)|^2} \right) \frac{\mathrm{d}w\, \mathrm{d}u}{|\gamma(u+w) - \gamma(u)|^2}$$

where

$$U_\varepsilon := \mathbb{R}/\mathbb{Z} \times \left([-\tfrac{1}{2}, -\varepsilon] \cup [\varepsilon, \tfrac{1}{2}] \right). \tag{1.2}$$

Though the space $H^{3/2} \cap H^{1,\infty}$ seems somewhat artificial at first sight, it just guarantees that we do not use bad parametrizations of our curves. The proof of this result is similar to [BR12].

We will then use the resulting Euler-Langrange equation for stationary points of the Möbius energy to prove that these points are smooth:

Theorem II (Stationary points are smooth). *Any stationary point $\gamma \in H^{\frac{3}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ of $E^{(2)}$, i.e., any curve $\gamma \in H^{\frac{3}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ for which*

$$\delta E^{(2)}(\gamma; h) = 0 \quad \text{for all } h \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n),$$

belongs to C^∞ when parametrized by arc-length.

In [BR12], improving a previous result [Rt12], the smoothness of stationary points of finite energy was already shown for the case of $E^{(\alpha)} := E^{(\alpha,1)}$, $\alpha \in (2,3)$, instead of the Möbius energy. It is worth noting, that those energies lead to a *subcritical* Euler-Langrage equation, and that in some sense the regularity theory can be based on Sobolev embeddings for fractional Sobolev and Besov spaces. In contrast to this, the Euler-Lagrange equation of the Möbius energy is *critical*. As for well-known critical geometric equations – like the Euler-Lagrange equation of the Willmore functional, see, e.g., [Sim93, Riv08], or harmonic maps on $\mathbb{R}^2$, see, e.g., [Hél91, Riv07], – one has first to find a way to gain an ε of additional regularity (via gaining a δ of additional integrability) and then start a bootstrapping argument. That is, in a quite natural way, the proof of Theorem II is an immediate consequence of two technically independent steps:

Theorem III (Initial regularity). *Let $\gamma \in H^{\frac{3}{2}}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$, $\gamma' \in \mathbb{S}^{n-1}$, be a stationary point of the Möbius energy, i.e. satisfying $\delta E^{(2)}(\gamma,\varphi) = 0$ for all $\varphi \in C^\infty(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$. Then $\gamma \in C^{1,\alpha}$, for some $\alpha > 0$, and $\gamma \in H^{\frac{3}{2},p}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ for some $p > 2$.*

Theorem IV (Bootstrapping). *For some $p > 2$, let $\gamma \in H^{\frac{3}{2},p}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$, $\gamma' \in \mathbb{S}^{n-1}$, be a stationary point of the Möbius energy. Then γ is smooth.*

Theorem III is proven in Section 3, Theorem IV in Section 4. While Theorem IV relies mainly on bringing together Sobolev embeddings and standard commutator estimates for Bessel potential spaces with techniques developed in [Bla12], some very delicate estimates are needed to get anything more than the critical and initial regularity $H^{\frac{3}{2}}$ for stationary points as stated in Theorem III.

Both theorems rely on a decomposition of the first variation dating back to [He00, Formula (4.5)] which already proved to be helpful in the analysis of the functionals $E^{(\alpha)}$ for $\alpha \in (2,3)$ (cf. [BR12]) and the gradient flow of the energies $E^{(\alpha)}$ for $\alpha \in [2,3)$ [Bla11a, Bla11b].

For $f, g : \mathbb{R} \to \mathbb{R}^n$, $\varepsilon > 0$ let

$$Q_\varepsilon(f,g) := \int_0^1 \int_{[-\frac{1}{2},\frac{1}{2}]\setminus(-\varepsilon,\varepsilon)} \Big(\langle f'(u), g'(u)\rangle\, w^2 - \langle f(u+w) - f(u), g(u+w) - g(u)\rangle \Big) \frac{dw}{w^4}\, du, \tag{1.3}$$

$$T_1(f,g) := -\int_0^1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \langle f'(u),\ g'(u)\rangle \left(\frac{1}{|f(u+w) - f(u)|^2} - \frac{1}{|w|^2} \right) dw\, du, \tag{1.4}$$

and

$$T_2(f,g) := \int_0^1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \langle f(u+w)-f(u),\ g(u+w)-g(u)\rangle \left(\frac{1}{|f(u+w)-f(u)|^4} - \frac{1}{|w|^4}\right) dw\, du. \tag{1.5}$$

From Theorem I we deduce that a critical knot $\gamma \in H^{\frac{3}{2}}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$, parametrized by arc length, satisfies

$$Q(\gamma,h) := \lim_{\varepsilon\to 0} Q_\varepsilon(\gamma,h) = T_1(\gamma,h) + T_2(\gamma,h) \quad \text{for all } h \in C^\infty(\mathbb{R}/\mathbb{Z}). \tag{1.6}$$

This is the form of the Euler-Lagrange equation, with which we will work in the proofs of Theorem III and Theorem IV.

Let us conclude this introduction by remarking that, in contrast to stationary points of $E^{(\alpha,1)}$, for $p > 1$ we do not expect stationary points of $E^{(\alpha,p)}$ to be C^∞-smooth: The resulting Euler-Lagrange equation should be in some sense a nonlocal degenerate elliptic equation. Keeping in mind the regularity theory for elliptic degenerate equations, one might expect nevertheless that stationary points are at least a bit more regular than an arbitrary finite-energy curve alone.

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2. Euler-Lagrange equation: Proof of Theorem I

This section is devoted to the proof of Theorem I which especially involves the derivation of a formula for the first variation.

By $H^{3/2}_{\mathrm{ir}}$ we will denote the set of injective and regular curves in $H^{3/2}$. The set $H^{1,\infty}_{\mathrm{ir}}$ is defined accordingly. First we will need the following lemma, to guarantee that $E^{(2)}$ is well defined on a sufficiently small $H^{3/2}\cap H^{1,\infty}$ neighborhood of the curve γ:

Lemma 2.1 ($H^{3/2}_{\mathrm{ir}}\cap H^{1,\infty}$ is open in $H^{3/2}\cap H^{1,\infty}$). *For any $\gamma \in H^{3/2}_{\mathrm{ir}}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)\cap H^{1,\infty}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ there is some $\tau_0 = \tau_0(\gamma) > 0$ with*

$$Y := \left\{\gamma+\varphi \,\middle|\, \varphi \in H^{1,\infty}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n), \left\|\varphi'\right\|_{L^\infty} \le \tau_0\right\} \subset H^{1,\infty}_{\mathrm{ir}}. \tag{2.1}$$

Moreover, there is a constant $c = c(\gamma) > 0$ with

$$\min\{|\tilde{\gamma}(u+w) - \tilde{\gamma}(u)|, d_{\tilde{\gamma}}(u+w,u)\} \geq c|w|, \qquad |\tilde{\gamma}'(u)| \geq c \tag{2.2}$$

for all $\tilde{\gamma} \in Y$ and $(u,w) \in U_0$.

Proof. We first show that γ is bi-Lipschitz. To this end, choose $\delta \in (0, \frac{1}{2})$ with

$$\left(\int\limits_{B_r(z)} \int\limits_{B_r(0)} \frac{|\gamma'(u+w) - \gamma'(u)|^2}{|w|^2} \, \mathrm{d}w \, \mathrm{d}u \right)^{1/2} \leq \tfrac{1}{2}$$

for all $z \in \mathbb{R}/\mathbb{Z}$ and all $r \in [0, \delta]$ which gives

$$\begin{aligned}
&\frac{1}{2r} \int\limits_{B_r(z)} \left| \gamma'(x) - \frac{1}{2r} \int\limits_{B_r(z)} \gamma'(y) \, \mathrm{d}y \right| \mathrm{d}x \\
&\leq \frac{1}{4r^2} \int\limits_{B_r(z)} \int\limits_{B_r(z)} |\gamma'(x) - \gamma'(y)| \, \mathrm{d}x \, \mathrm{d}y \\
&\leq \left(\frac{1}{4r^2} \int\limits_{B_r(z)} \int\limits_{B_r(z)} |\gamma'(x) - \gamma'(y)|^2 \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2} \\
&\leq \left(\int\limits_{B_r(z)} \int\limits_{B_r(z)} \frac{|\gamma'(x) - \gamma'(y)|^2}{|x-y|^2} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2} \\
&\leq \tfrac{1}{2}.
\end{aligned}$$

Since $\left| \frac{1}{2r} \int_{B_r(z)} \gamma'(y) \, \mathrm{d}y \right| \leq 1$ we deduce that

$$\inf_{\substack{a \in \mathbb{R}^n \\ |a| \leq 1}} \frac{1}{2r} \int\limits_{B_r(z)} \left| \gamma'(y) - a \right| \mathrm{d}y \leq \tfrac{1}{2}.$$

For $x, y \in \mathbb{R}/\mathbb{Z}$ with $|x-y| \leq 2\delta$ let $r := \frac{1}{2}|x-y|$ and $z \in \mathbb{R}/\mathbb{Z}$ be the midpoint of the shorter arc between x and y. Then

$$\begin{aligned}
|\gamma(x) - \gamma(y)| &= \sup_{\substack{a \in \mathbb{R}^n \\ |a| \leq 1}} \int\limits_{B_r(z)} \langle \gamma'(t), a \rangle \, \mathrm{d}t \\
&= \sup_{\substack{a \in \mathbb{R}^n \\ |a| \leq 1}} \int\limits_{B_r(z)} \langle \gamma'(t), \gamma'(t) + (a - \gamma'(t)) \rangle \, \mathrm{d}t
\end{aligned}$$

$$\geq \left(1 - \inf_{\substack{a\in\mathbb{R}^n \\ |a|\leq 1}} \frac{1}{2r} \int_{B_r(z)} |\gamma'(t) - a| \, \mathrm{d}t\right) |x - y|$$
$$\geq \tfrac{1}{2}|x - y|$$

for all $x, y \in \mathbb{R}/\mathbb{Z}$ with $|x - y| \leq 2\delta$. Since γ is embedded and

$$(x, y) \mapsto \frac{|\gamma(y) - \gamma(x)|}{|y - x|}$$

defines a continuous positive function on $I_\delta := \{(x, y) \in (\mathbb{R}/\mathbb{Z})^2 : |x - y| \geq 2\delta\}$, we furthermore have

$$|\gamma(x) - \gamma(y)| \geq \underbrace{\min_{(\tilde{x},\tilde{y})\in I_\delta} \frac{|\gamma(\tilde{y}) - \gamma(\tilde{x})|}{|\tilde{y} - \tilde{x}|}}_{>0} |x - y|.$$

for all $(x, y) \in I_\delta$. Hence, there is a $c_0 = c_0(\gamma) > 0$ with

$$|\gamma(x) - \gamma(x + w)| \geq c_0|w|$$

for all $w \in [-1/2, 1/2]$. Lessening c_0 if necessary, we can also achieve by regularity

$$\left|\gamma'\right| \geq c_0 \qquad \text{on } \mathbb{R}/\mathbb{Z}.$$

Letting $\tau_0 := \frac{1}{2}c_0$ we obtain for arbitrary $\tilde{\gamma} \in Y$

$$\begin{aligned} |\tilde{\gamma}(u + w) - \tilde{\gamma}(u)| &\geq |\gamma(u + w) - \gamma(u)| - |(\tilde{\gamma} - \gamma)(u + w) - (\tilde{\gamma} - \gamma)(u)| \\ &\geq c_0|w| - \left\|\gamma' - \tilde{\gamma}'\right\|_{L^\infty} |w| \\ &\geq \tfrac{1}{2}c_0|w| \end{aligned}$$

and

$$\left|\tilde{\gamma}'\right| \geq \left|\gamma'\right| - \left|\tilde{\gamma}' - \gamma'\right| \geq c_0 - \left\|\gamma' - \tilde{\gamma}'\right\|_{L^\infty} \geq \tfrac{1}{2}c_0.$$

From the latter estimate we deduce by (1.1) for $u \in \mathbb{R}/\mathbb{Z}$, $w \in [-\frac{1}{2}, \frac{1}{2}]$

$$d_{\tilde{\gamma}}(u + w, u) \geq d_{\tilde{\gamma}}(u \pm w, u) \geq \tfrac{1}{2}c_0|w|.$$

We have established (2.2) which gives (2.1). □

We will use the last lemma to prove the following theorem, from which Theorem I will follow quite easily.

Proposition 2.2. *The energy $E^{(2)}$ is continuously differentiable on $\left(H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}\right)(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. The derivative of $E^{(2)}$ at $\gamma \in H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}$ in direction $\varphi \in H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}$ exists and is given by*

$$\begin{aligned}\delta E^{(2)}(\gamma;\varphi) = \\ 2\lim_{\varepsilon \searrow 0} \iint\limits_{U_\varepsilon} \Bigg\{ \left(\frac{1}{|\gamma(u+w)-\gamma(u)|^2} - \frac{1}{d_\gamma(u+w,u)^2} \right) \left\langle \frac{\gamma'(u)}{|\gamma'(u)|^2}, \varphi'(u) \right\rangle \\ - \left(\frac{\langle \gamma(u+w)-\gamma(u), \varphi(u+w)-\varphi(u)\rangle}{|\gamma(u+w)-\gamma(u)|^4} - \frac{\frac{\mathrm{d}}{\mathrm{d}\tau}\big|_{\tau=0} d_{\gamma+\tau\varphi}(u+w,u)}{d_\gamma(u+w,u)^3} \right) \Bigg\} \\ |\gamma'(u+w)||\gamma'(u)| \,\mathrm{d}w \,\mathrm{d}u.\end{aligned} \tag{2.3}$$

As γ is absolutely continuous and regular, the derivative $\frac{\mathrm{d}}{\mathrm{d}\tau}\big|_{\tau=0} d_{\gamma+\tau\varphi}(u+w,u)$ is well-defined for almost all $(u,w) \in U_0$. From (1.1) we deduce

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\big|_{\tau=0} d_{\gamma+\tau\varphi}(u+w,u) = \begin{cases} |w| \int_0^1 \left\langle \frac{\gamma'(u+\sigma w)}{|\gamma'(u+\sigma w)|}, \varphi'(u+\sigma w) \right\rangle d\sigma & \text{if } \mathscr{L}(\gamma|_{[u,u+w]}) < \frac{1}{2}\mathscr{L}(\gamma), \\ -|w| \int_0^1 \left\langle \frac{\gamma'(u+\sigma w)}{|\gamma'(u+\sigma w)|}, \varphi'(u+\sigma w) \right\rangle d\sigma & \text{if } \mathscr{L}(\gamma|_{[u,u+w]}) > \frac{1}{2}\mathscr{L}(\gamma). \end{cases} \tag{2.4}$$

To prove Proposition 2.2, we will first show that the following approximations of the energy $E^{(2)}$, in which we cut off the singular part, are continuously differentiable and provide a formula for the first variation. For $\varepsilon \in (0, \frac{1}{2})$ we set

$$E^{(2)}_\varepsilon(\gamma) := \iint\limits_{U_\varepsilon} \left(\frac{1}{|\gamma(u+w)-\gamma(u)|^2} - \frac{1}{d_\gamma(u+w,u)^2} \right) \big|\gamma'(u+w)\big|\gamma'(u)\big| \,\mathrm{d}w \,\mathrm{d}u.$$

Lemma 2.3. *For $\varepsilon \in (0, \frac{1}{2})$ the functional $E^{(2)}_\varepsilon$ is continuously differentiable on the space of all injective regular curves in $H^{3/2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \cap H^{1,\infty}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. The directional derivative at $\gamma \in H^{3/2}_{\mathrm{ir}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \cap H^{1,\infty}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ in direction $\varphi \in H^{3/2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \cap H^{1,\infty}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ is given by*

$$\begin{aligned}\delta E^{(2)}_\varepsilon(\gamma;\varphi) = \\ 2 \iint\limits_{U_\varepsilon} \Bigg\{ \left(\frac{1}{|\gamma(u+w)-\gamma(u)|^2} - \frac{1}{d_\gamma(u+w,u)^2} \right) \left\langle \frac{\gamma'(u)}{|\gamma'(u)|^2}, \varphi'(u) \right\rangle \\ - \left(\frac{\langle \gamma(u+w)-\gamma(u), \varphi(u+w)-\varphi(u)\rangle}{|\gamma(u+w)-\gamma(u)|^4} - \frac{\frac{\mathrm{d}}{\mathrm{d}\tau}\big|_{\tau=0} d_{\gamma+\tau\varphi}(u+w,u)}{d_\gamma(u+w,u)^3} \right) \Bigg\} \\ |\gamma'(u+w)||\gamma'(u)| \,\mathrm{d}w \,\mathrm{d}u.\end{aligned} \tag{2.5}$$

Proof. Applying Lemma 2.1, we obtain an $H^{3/2} \cap H^{1,\infty}$-neighborhood $Y \subset H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}$ of γ such that (2.2) uniformly holds on Y for any element in U_ε. The integrand in (2.5) is almost everywhere the pointwise derivative of the integrand in $E^{(2)}_\varepsilon$. Using (1.1) and (2.4), one sees furthermore that this pointwise derivative is majorized by some L^1-function. So, Lebesgue's Theorem permits to interchange differentiation and integration which, by a suitable reparametrization, results in (2.5).

As for continuity of $E^{(2)}_\varepsilon$ and $\delta E^{(2)}_\varepsilon$ the only difficulty is to treat the intrinsic distance. Recalling the continuity of the length functional with respect to absolutely continuous curves we can directly read off from (1.1) that the integrand of $E^{(2)}_\varepsilon$ defines a continuous operator $(H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty})(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \to L^1(U_\varepsilon)$.

Since γ is regular, for any $u \in \mathbb{R}/\mathbb{Z}$ there are at most two points $w \in [-\frac{1}{2}, \frac{1}{2}]$ satisfying $\mathscr{L}(\gamma|_{[u,u+w]}) = \frac{1}{2}\mathscr{L}(\gamma)$ which results in a null set in U_0. Additionally using (2.4), we see that the integrand of $\delta E^{(2)}_\varepsilon$ gives rise to a continuous mapping $(H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty})(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \times H^{1,\infty}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \to L^1(U_\varepsilon)$. Being linear and bounded in the second component, it can be viewed as a continuous mapping from $H^{1,\infty}$ into the linear bounded operators $H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty} \to L^1(U_\varepsilon)$.

Altogether, the integrand of $E^{(2)}_\varepsilon$ is a continuously differentiable functional $H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty} \to L^1(U_\varepsilon)$. The statement now follows from the chain rule and the fact that the integration operator

$$L^1(U_\varepsilon) \to \mathbb{R}, \qquad g \mapsto \iint_{U_\varepsilon} g(u, w)\,\mathrm{d}u\,\mathrm{d}w,$$

is continuously differentiable as it is a bounded linear operator. □

Due to the fact that bounded L^1-sequences are not uniformly integrable, the approximations $E^{(2)}_\varepsilon$ do not even form a Cauchy sequence in $C^0(H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty})$. In order to prove Proposition 2.2, we state in Lemma 2.4 below that $E^{(2)}_\varepsilon$ is *nearly* a Cauchy sequence in $C^1(X_\delta)$ for subsets $X_\delta \subset H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}$, $\delta \geq 0$, which satisfy the following substitute of the uniform integrability property

$$\limsup_{\varepsilon \to 0} \sup_{\gamma \in X_\delta} \left(\iint_{\mathbb{R}/\mathbb{Z} \times [-\varepsilon, \varepsilon]} \frac{|\gamma'(u+w) - \gamma'(u)|^2}{w^2}\,\mathrm{d}w\,\mathrm{d}u \right)^{1/2} \leq \delta. \tag{2.6}$$

The statement involves the Lipschitz constant

$$\operatorname{lip}_Y E = \sup_{\substack{f, \tilde{f} \in Y \\ f \neq \tilde{f}}} \frac{\left|E(f) - E(\tilde{f})\right|}{\left\|f - \tilde{f}\right\|}$$

for some real-valued functional E and a subset Y contained in its domain.

Lemma 2.4. *We have*

$$E^{(2)}_\varepsilon(\gamma) \to E^{(2)}(\gamma)$$

for all $\gamma \in H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}$.

Furthermore, for any $\gamma_0 \in H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}$ *there is an open subset* $Y \subset H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}$ *and a constant* $C = C(\gamma_0) < \infty$ *such that*

$$\limsup_{\varepsilon_1,\varepsilon_2 \to 0} \operatorname{lip}_{X_\delta \cap Y}(E^{(2)}_{\varepsilon_1} - E^{(2)}_{\varepsilon_2}) \le C\delta \tag{2.7}$$

for all subsets $X_\delta \subset H^{3/2} \cap H^{1,\infty}$ *satisfying* (2.6) *with* $\delta \in [0,1]$.

Proof. From Lemma 2.1 we get a $H^{3/2} \cap H^{1,\infty}$-neighborhood $Y \subset H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}$ of γ such that (2.2) holds for all $\gamma \in Y$. Making Y smaller if necessary, we may also assume the existence of an $\varepsilon_0 > 0$ with

$$d_\gamma(u+w,u) = \mathscr{L}(\gamma|_{[u,u+w]})$$

for all $\gamma \in Y$ and $w \in [-\varepsilon_0, \varepsilon_0]$.

In order to bring the integrand in the definition of $E^{(2)}$ and $E^{(2)}_\varepsilon$ in a more convenient form we introduce the function

$$g(\zeta,\eta,\vartheta,\iota) := \frac{\zeta^{-2} - \eta^{-2}}{\eta^2 - \zeta^2}\vartheta\iota$$

which is Lipschitz continuous and positive on $[\tilde{c},\infty)^4$ for any $\tilde{c} > 0$. We define for $u \in \mathbb{R}/\mathbb{Z}$, $w \in [-\varepsilon_0, \varepsilon_0]$

$$\mathcal{G}_\gamma : (u,w) \mapsto g\left(\left|\int_0^1 \gamma'(u+\theta_1 w)\,\mathrm{d}\theta_1\right|, \int_0^1 \left|\gamma'(u+\theta_2 w)\right|\mathrm{d}\theta_2, \left|\gamma'(u+w)\right|, \left|\gamma'(u)\right|\right).$$

We have chosen Y in such a way that the arguments in $\mathcal{G}$ are uniformly bounded away from zero. Then we decompose the integrand in the definition of $E^{(2)}$ for $|w| \le \varepsilon_0$ into

$$\begin{aligned}
&\left(\frac{1}{|\gamma(u+w)-\gamma(u)|^2} - \frac{1}{d_\gamma(u+w,u)^2}\right)\left|\gamma'(u+w)\right|\left|\gamma'(u)\right| \\
&= \frac{1}{w^2}\left(\frac{1}{\left|\int_0^1 \gamma'(u+\theta_1 w)\,\mathrm{d}\theta_1\right|^2} - \frac{1}{\left(\int_0^1 |\gamma'(u+\theta_2 w)|\,\mathrm{d}\theta_2\right)^2}\right)\left|\gamma'(u+w)\right|\left|\gamma'(u)\right| \\
&= \mathcal{G}_\gamma(u,w)\frac{\left(\int_0^1 |\gamma'(u+\theta_2 w)|\,\mathrm{d}\theta_2\right)^2 - \left|\int_0^1 \gamma'(u+\theta_1 w)\,\mathrm{d}\theta_1\right|^2}{|w|^2} \\
&= \mathcal{G}_\gamma(u,w)\frac{\iint_{[0,1]^2}\left(|\gamma'(u+\theta_1 w)||\gamma'(u+\theta_2 w)| - \langle\gamma'(u+\theta_1 w),\gamma'(u+\theta_2 w)\rangle\right)\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{|w|^2}.
\end{aligned}$$

Using $2|a||b| - 2\langle a, b\rangle = |a-b|^2 - ||a|-|b||^2$ for $a, b \in \mathbb{R}^n$ this can be written as

$$\begin{aligned}&\tfrac{1}{2}\mathcal{G}_\gamma(u,w)\frac{\iint_{[0,1]^2}|\gamma'(u+\theta_1 w)-\gamma'(u+\theta_2 w)|^2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{w^2}\\&\quad-\tfrac{1}{2}\mathcal{G}_\gamma(u,w)\frac{\iint_{[0,1]^2}\left(|\gamma'(u+\theta_1 w)|-|\gamma'(u+\theta_2 w)|\right)^2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{w^2}.\end{aligned}$$

We first use this to get

$$\begin{aligned}E^{(2)}(\gamma)-E^{(2)}_\varepsilon(\gamma)&\le C\int_{\mathbb{R}/\mathbb{Z}}\int_{-\varepsilon}^{\varepsilon}\mathcal{G}_\gamma(u,w)\frac{\iint_{[0,1]^2}|\gamma'(u+\theta_1 w)-\gamma'(u+\theta_2 w)|^2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{w^2}dwdu\\&\le C\int_{\mathbb{R}/\mathbb{Z}}\int_{-\varepsilon}^{\varepsilon}\frac{\iint_{[0,1]^2}|\gamma'(u+\theta_1 w)-\gamma'(u+\theta_2 w)|^2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{w^2}dwdu\xrightarrow{\varepsilon\to 0}0\end{aligned}$$

which proves the pointwise convergence stated in the lemma.

Let now $0<\varepsilon_1<\varepsilon_2<\varepsilon_0$ and set

$$F := E^{(2)}_{\varepsilon_1}-E^{(2)}_{\varepsilon_2}.$$

Using the decomposition of the integrand above, we get

$$\begin{aligned}F(\gamma)&=\tfrac{1}{2}\int_{\mathbb{R}/\mathbb{Z}}\int_{\varepsilon_1<|w|<\varepsilon_2}\mathcal{G}_\gamma(u,w)\frac{\iint_{[0,1]^2}|\gamma'(u+\theta_1 w)-\gamma'(u+\theta_2 w)|^2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{w^2}\,\mathrm{d}w\,\mathrm{d}u\\&\quad-\tfrac{1}{2}\int_{\mathbb{R}/\mathbb{Z}}\int_{\varepsilon_1<|w|<\varepsilon_2}\mathcal{G}_\gamma(u,w)\frac{\iint_{[0,1]^2}\left(|\gamma'(u+\theta_1 w)|-|\gamma'(u+\theta_2 w)|\right)^2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{w^2}\,\mathrm{d}w\,\mathrm{d}u\\&=:\tfrac{1}{2}F^{(2)}_1(\gamma)-\tfrac{1}{2}F^{(2)}_2(\gamma).\end{aligned}$$

To estimate the difference $F(\tilde\gamma)-F(\gamma)$ for $\gamma, \tilde\gamma \in Y_0$, we first consider

$$\begin{aligned}&\left|\mathcal{G}_{\tilde\gamma}(u,w)-\mathcal{G}_\gamma(u,w)\right|\\&\le C\left|\left|\int_0^1\tilde\gamma'(u+\theta_1 w)\,\mathrm{d}\theta_1\right|-\left|\int_0^1\gamma'(u+\theta_2 w)\,\mathrm{d}\theta_2\right|\right|\\&\quad+C\left|\int_0^1\left(\left|\tilde\gamma'(u+\theta w)\right|-\left|\gamma'(u+\theta w)\right|\right)\mathrm{d}\theta\right|\\&\quad+C\left|\left|\tilde\gamma'(u+w)\right|-\left|\gamma'(u+w)\right|\right|+C\left|\left|\tilde\gamma'(u)\right|-\left|\gamma'(u)\right|\right|\end{aligned}$$

$$\leq C \int_0^1 \left|\tilde{\gamma}'(u+\theta w) - \gamma'(u+\theta w)\right| \mathrm{d}\theta + C\left|\tilde{\gamma}'(u+w) - \gamma'(u+w)\right| + C\left|\tilde{\gamma}'(u) - \gamma'(u)\right|$$
$$\leq C \left\|\tilde{\gamma}' - \gamma'\right\|_{L^\infty}.$$

We arrive at

$$\begin{aligned}
&\left|F_1^{(2)}(\tilde{\gamma}) - F_1^{(2)}(\gamma)\right| \\
&\leq \iint\limits_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon_2,\varepsilon_2]} \left|\mathcal{G}_{\tilde{\gamma}}(u,w) - \mathcal{G}_{\gamma}(u,w)\right| \frac{\iint_{[0,1]^2} |\tilde{\gamma}'(u+\theta_1 w) - \tilde{\gamma}'(u+\theta_2 w)|^2 \,\mathrm{d}\theta_1 \,\mathrm{d}\theta_2}{w^2} \,\mathrm{d}w \,\mathrm{d}u \\
&\quad + \iint\limits_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon_2,\varepsilon_2]} \left|\mathcal{G}_{\gamma}(u,w)\right| \\
&\qquad \frac{\iint_{[0,1]^2} \left||\tilde{\gamma}'(u+\theta_1 w) - \tilde{\gamma}'(u+\theta_2 w)|^2 - |\gamma'(u+\theta_1 w) - \gamma'(u+\theta_2 w)|^2\right| \mathrm{d}\theta_1 \,\mathrm{d}\theta_2}{w^2} \\
&\qquad\qquad \mathrm{d}w \,\mathrm{d}u \\
&\leq C \left\|\tilde{\gamma}' - \gamma'\right\|_{L^\infty} \iint\limits_{[0,1]^2} \iint\limits_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon_2,\varepsilon_2]} \frac{|\tilde{\gamma}'(u+\theta_1 w) - \tilde{\gamma}'(u+\theta_2 w)|^2}{w^2} \,\mathrm{d}w \,\mathrm{d}u \,\mathrm{d}\theta_1 \,\mathrm{d}\theta_2 \\
&\quad + C \left\|\gamma'\right\|_{L^\infty} \iint\limits_{[0,1]^2} \iint\limits_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon_2,\varepsilon_2]} \\
&\qquad \frac{|(\tilde{\gamma}' + \gamma')(u+\theta_1 w) - (\tilde{\gamma}' + \gamma')(u+\theta_2 w)||(\tilde{\gamma}' - \gamma')(u+\theta_1 w) - (\tilde{\gamma}' - \gamma')(u+\theta_2 w)|}{w^2} \\
&\qquad\qquad \mathrm{d}w \,\mathrm{d}u \,\mathrm{d}\theta_1 \,\mathrm{d}\theta_2 \\
&\leq C \left[\tilde{\gamma}'\right]_{H^{1/2}_{2\varepsilon_2}}^2 \left\|\tilde{\gamma}' - \gamma'\right\|_{L^\infty} + C \left\|\gamma'\right\|_{L^\infty} \left[\tilde{\gamma}' + \gamma'\right]_{H^{1/2}_{2\varepsilon_2}} \left[\tilde{\gamma}' - \gamma'\right]_{H^{1/2}_{2\varepsilon_2}}
\end{aligned}$$

where we set for $\varepsilon \in (0, \frac{1}{2})$

$$[f]_{H^{1/2}_{\varepsilon}} := \left(\iint_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon,\varepsilon]} \frac{|f(u+w) - f(u)|^2}{w^2} \,\mathrm{d}w \,\mathrm{d}u \right)^{1/2}.$$

For the second term we compute

$$\begin{aligned}
&\left|F_2^{(2)}(\tilde{\gamma}) - F_2^{(2)}(\gamma)\right| \\
&\leq \iint\limits_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon_2,\varepsilon_2]} \left|\mathcal{G}_{\tilde{\gamma}}(u,w) - \mathcal{G}_{\gamma}(u,w)\right| \frac{\iint_{[0,1]^2} (|\tilde{\gamma}'(u+\theta_1 w)| - |\tilde{\gamma}'(u+\theta_2 w)|)^2 \,\mathrm{d}\theta_1 \,\mathrm{d}\theta_2}{w^2} \,\mathrm{d}w \,\mathrm{d}u
\end{aligned}$$

$$+ \iint\limits_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon_2,\varepsilon_2]} \left|\mathcal{G}_\gamma(u,w)\right| \cdot \cdot \frac{\iint_{[0,1]^2} \left|(|\tilde\gamma'(u+\theta_1 w)| - |\tilde\gamma'(u+\theta_2 w)|)^2 - (|\gamma'(u+\theta_1 w)| - |\gamma'(u+\theta_2 w)|)^2\right| \mathrm{d}\theta_1\, \mathrm{d}\theta_2}{w^2} \mathrm{d}w\, \mathrm{d}u$$

$$\leq C \left\|\tilde\gamma' - \gamma'\right\|_{L^\infty} \iint\limits_{[0,1]^2} \iint\limits_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon_2,\varepsilon_2]} \frac{|\tilde\gamma'(u+\theta_1 w) - \tilde\gamma'(u+\theta_2 w)|^2}{w^2} \mathrm{d}w\, \mathrm{d}u\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2$$

$$+ C \left\|\gamma'\right\|_{L^\infty} \iint\limits_{[0,1]^2} \iint\limits_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon_2,\varepsilon_2]} \frac{|(|\tilde\gamma'(u+\theta_1 w)| - |\tilde\gamma'(u+\theta_2 w)|) + (|\gamma'(u+\theta_1 w)| - |\gamma'(u+\theta_2 w)|)|}{|w|} \cdot$$

$$\cdot \frac{|(|\tilde\gamma'(u+\theta_1 w)| - |\tilde\gamma'(u+\theta_2 w)|) - (|\gamma'(u+\theta_1 w)| - |\gamma'(u+\theta_2 w)|)|}{|w|} \mathrm{d}w\, \mathrm{d}u\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2$$

$$\leq C \left\|\tilde\gamma' - \gamma'\right\|_{L^\infty} [\tilde\gamma']^2_{H^{1/2}_{2\varepsilon_2}} + C \left\|\gamma'\right\|_{L^\infty} \left[|\tilde\gamma'| + |\gamma'|\right]_{H^{1/2}_{2\varepsilon_2}} \left[|\tilde\gamma'| - |\gamma'|\right]_{H^{1/2}_{2\varepsilon_2}}.$$

Using the chain and product rule for Sobolev spaces and the formula

$$\left|\tilde\gamma'\right| - \left|\gamma'\right| = \frac{\langle \tilde\gamma' + \gamma', \tilde\gamma' - \gamma'\rangle}{|\tilde\gamma'| + |\gamma'|},$$

we obtain, assuming $\varepsilon_0 < \frac{1}{4}$, for $C > 0$ depending on the constant from (2.2)

$$\left[|\tilde\gamma'| - |\gamma'|\right]_{H^{1/2}_{2\varepsilon_2}} \leq \left[|\tilde\gamma'| - |\gamma'|\right]_{H^{1/2}(\mathbb{R}/\mathbb{Z})}$$

$$\leq C \left\|\tilde\gamma' - \gamma'\right\|_{H^{1/2}\cap L^\infty} \left\|\tilde\gamma' + \gamma'\right\|_{H^{1/2}\cap L^\infty} \left(1 + [\tilde\gamma']_{H^{1/2}_{2\varepsilon_2}} + [\gamma']_{H^{1/2}_{2\varepsilon_2}}\right)$$

where

$$\|\cdot\|_{H^{1/2}\cap L^\infty} := \|\cdot\|_{H^{1/2}} + \|\cdot\|_{L^\infty}.$$

Hence

$$\left|F_2^{(2)}(\tilde\gamma) - F_2^{(2)}(\gamma)\right| \leq C\Bigg([\tilde\gamma']^2_{H^{1/2}_{2\varepsilon_2}} + \left\|\gamma'\right\|_{L^\infty} \left([\tilde\gamma']_{H^{1/2}_{2\varepsilon_2}} + [\gamma']_{H^{1/2}_{2\varepsilon_2}}\right) \left\|\tilde\gamma' + \gamma'\right\|_{H^{1/2}\cap L^\infty} \cdot$$

$$\cdot \left(1 + [\tilde\gamma']_{H^{1/2}_{2\varepsilon_2}} + [\gamma']_{H^{1/2}_{2\varepsilon_2}}\right)\Bigg) \|\tilde\gamma' - \gamma'\|_{H^{1/2}\cap L^\infty}.$$

The claim follows from

$$\limsup_{\varepsilon_2 \searrow 0} \sup_{\tilde\gamma \in X_\delta \cap Y} [\tilde\gamma']_{H^{1/2}_{2\varepsilon_2}} \leq \delta.$$

□

Proof of Proposition 2.2. In order to prove that *directional derivatives* exist at $\gamma_0 \in H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}$ for all directions $\varphi \in H^{3/2} \cap H^{1,\infty}$ let Y be as in the proof of Lemma 2.4 and

$$X_0 := \{\gamma_0 + \tau\varphi \,|\, \tau \in (-1,1)\}\,.$$

First we observe that X_0 satisfies (2.6), thus being an admissible set for Lemma 2.4. Indeed, for $\gamma_\tau := \gamma_0 + \tau\varphi$, $|\tau| \le 1$,

$$\begin{aligned}
&\left(\iint_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon,\varepsilon]} \frac{|\gamma_\tau'(u+w)-\gamma_\tau'(u)|^2}{|w|^2}\,\mathrm{d}w\,\mathrm{d}u\right)^{1/2}\\
&\le \left(\iint_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon,\varepsilon]} \frac{|\gamma_0'(u+w)-\gamma_0'(u)|^2}{|w|^2}\,\mathrm{d}w\,\mathrm{d}u\right)^{1/2} + \left(\iint_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon,\varepsilon]} \frac{|\varphi'(u+w)-\varphi'(u)|^2}{|w|^2}\,\mathrm{d}w\,\mathrm{d}u\right)^{1/2}\\
&\xrightarrow{\varepsilon\searrow 0} 0. \qquad (2.8)
\end{aligned}$$

From this we deduce, for

$$f_\varepsilon : \tau \mapsto E^{(2)}_\varepsilon(\gamma_0 + \tau\varphi),$$

that

$$\begin{aligned}
&\left|f_{\varepsilon_1}'(\tau) - f_{\varepsilon_2}'(\tau)\right| = \left|\delta E^{(2)}_{\varepsilon_1}(\gamma_0+\tau\varphi;\varphi) - \delta E^{(2)}_{\varepsilon_2}(\gamma_0+\tau\varphi;\varphi)\right|\\
&\le \limsup_{\theta\to 0}\left|\frac{E^{(2)}_{\varepsilon_1}(\gamma_0+(\tau+\theta)\varphi) - E^{(2)}_{\varepsilon_1}(\gamma_0+\tau\varphi)}{\theta} - \frac{E^{(2)}_{\varepsilon_2}(\gamma_0+(\tau+\theta)\varphi) - E^{(2)}_{\varepsilon_2}(\gamma_0+\tau\varphi)}{\theta}\right|\\
&\le \mathrm{lip}_{X_0\cap Y}\left(E^{(2)}_{\varepsilon_1} - E^{(2)}_{\varepsilon_2}\right)\|\varphi\|_{H^{3/2}\cap H^{1,\infty}} \qquad (2.9)\\
&\xrightarrow{\varepsilon_1,\varepsilon_2\searrow 0} 0 \qquad \text{by (2.7).}
\end{aligned}$$

As $E^{(2)}_\varepsilon \to E^{(2)}$ pointwise, this proves that $(f_\varepsilon)_{\varepsilon>0}$ is a Cauchy sequence in $C^1((-\tau_0,\tau_0))$ converging to $E^{(2)}(\gamma_0+\tau\varphi) = \lim_{\varepsilon\searrow 0} E^{(2)}_\varepsilon(\gamma_0+\tau\varphi)$ as $\varepsilon \to 0$. Hence, especially all directional derivatives of $E^{(2)}$ exist and

$$\delta E^{(2)}(\gamma_0;\varphi) = \lim_{\varepsilon\searrow 0} \delta E^{(2)}_\varepsilon(\gamma_0;\varphi)$$

for all $\gamma_0 \in H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}$, $\varphi \in H^{3/2} \cap H^{1,\infty}$.

The next step is to establish *Gâteaux differentiability*. To this end we merely have to show $\delta E^{(2)}(\gamma_0,\cdot) \in \left(H^{3/2}\cap H^{1,\infty}\right)^*$ for $\gamma_0 \in H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}$. Linearity carries over from $E^{(2)}_\varepsilon$. In order to prove boundedness we introduce

$$X_\delta := \{\gamma \in H^{3/2} : \|\gamma - \gamma_0\|_{H^{3/2}} \le \delta\} \qquad \text{for } \delta \in (0,1] \qquad (2.10)$$

which also satisfies (2.6) as for $\gamma \in X_\delta$ we have

$$\begin{aligned}&\left(\iint_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon,\varepsilon]} \frac{|\gamma'(u+w)-\gamma'(u)|^2}{|w|^2}\,\mathrm{d}w\,\mathrm{d}u\right)^{1/2}\\ &\le \left(\iint_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon,\varepsilon]} \frac{|\gamma_0'(u+w)-\gamma_0'(u)|^2}{|w|^2}\,\mathrm{d}w\,\mathrm{d}u\right)^{1/2} + \delta \xrightarrow{\varepsilon\searrow 0} \delta.\end{aligned} \tag{2.11}$$

Now

$$\begin{aligned}\delta E^{(2)}(\gamma_0;\varphi) &= \delta E^{(2)}_\varepsilon(\gamma_0;\varphi) + \delta E^{(2)}(\gamma_0;\varphi) - \delta E^{(2)}_\varepsilon(\gamma_0;\varphi)\\ &= \delta E^{(2)}_\varepsilon(\gamma_0;\varphi) + \lim_{\varepsilon_1\searrow 0}\left(\delta E^{(2)}_{\varepsilon_1}(\gamma_0;\varphi) - \delta E^{(2)}_\varepsilon(\gamma_0;\varphi)\right)\end{aligned}$$

and thus, arguing as in (2.9) and recalling $\delta E^{(2)}_\varepsilon(\gamma_0;\cdot) \in \left(H^{3/2}\cap H^{1,\infty}\right)^*$,

$$|\delta E^{(2)}(\gamma_0;\varphi)| \le |\delta E^{(2)}_\varepsilon(\gamma_0;\varphi)| + \underbrace{\limsup_{\varepsilon_1\searrow 0} \mathrm{lip}_{X_\delta\cap Y}\left(E^{(2)}_{\varepsilon_1} - E^{(2)}_\varepsilon\right)}_{<\infty} \|\varphi\|_{H^{3/2}\cap H^{1,\infty}} \le C\|\varphi\|_{H^{3/2}\cap H^{1,\infty}}$$

for all $\gamma_0 \in H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}$ and $\varphi \in H^{3/2}\cap H^{1,\infty}$. Hence, $E^{(2)}$ is Gâteaux differentiable and the differential is given by

$$\left(E^{(2)}\right)'(\gamma_0) = \delta E^{(2)}(\gamma_0;\cdot)$$

for all $\gamma_0 \in H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}$, $\varphi \in H^{3/2}\cap H^{1,\infty}$.

Finally, to see that the differential is *continuous*, let $\sigma > 0$ be given and let us choose $\delta > 0$ and $\varepsilon > 0$ so small that

$$\mathrm{lip}_{X_\delta\cap Y}\left(E^{(2)}_{\varepsilon_1} - E^{(2)}_{\varepsilon_2}\right) \overset{(2.7)}{\le} C\delta \le \tfrac{1}{3}\sigma$$

for all $\varepsilon_1, \varepsilon_2 < \varepsilon$ where X_δ is as in (2.10). Then we have for $\gamma \in X_\delta \cap Y$ and any $\varphi \in H^{3/2}\cap H^{1,\infty}$

$$\begin{aligned}|\delta E^{(2)}(\gamma;\varphi) - \delta E^{(2)}(\gamma_0;\varphi)| &\le |\delta E^{(2)}(\gamma;\varphi) - \delta E^{(2)}_\varepsilon(\gamma;\varphi)| + |\delta E^{(2)}_\varepsilon(\gamma;\varphi) - \delta E^{(2)}_\varepsilon(\gamma_0;\varphi)|\\ &\quad + |\delta E^{(2)}_\varepsilon(\gamma_0;\varphi) - \delta E^{(2)}(\gamma_0;\varphi)|\\ &\overset{(2.9)}{\le} |\delta E^{(2)}_\varepsilon(\gamma;\varphi) - \delta E^{(2)}_\varepsilon(\gamma_0;\varphi)| + \tfrac{2}{3}\sigma\|\varphi\|_{H^{3/2}\cap H^{1,\infty}}.\end{aligned}$$

Since $E^{(2)}_\varepsilon$ is C^1 we deduce that there is an open neighborhood $V \subset X_\delta$ of γ_0 such that

$$|\delta E^{(2)}_\varepsilon(\gamma;\varphi) - \delta E^{(2)}_\varepsilon(\gamma_0;\varphi)| \le \tfrac{1}{3}\sigma\|\varphi\|_{H^{3/2}\cap H^{1,\infty}}$$

and hence

$$|\delta E^{(2)}(\gamma;\varphi) - \delta E^{(2)}(\gamma_0;\varphi)| \le \sigma\|\varphi\|_{H^{3/2}\cap H^{1,\infty}}$$

for all $\gamma \in V$. This proves that $\left(E^{(2)}\right)'$ is continuous from $H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}$ into $\left(H^{3/2}\cap H^{1,\infty}\right)^*$ and hence $E^{(2)}$ is $C^1(H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty})$. □

Proof of Theorem I. Using Proposition 2.2 we merely have to derive the formula of the first variation for a curve $\gamma \in H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}$ parametrized by arc-length and $\varphi \in H^{3/2} \cap H^{1,\infty}$. As $|\gamma'| \equiv 1$ a. e. we deduce from Lemma 2.3 and Equation (2.4)

$$\delta E^{(2)}(\gamma;\varphi) \overset{\varepsilon\searrow 0}{\longleftarrow} 2 \iint_{U_\varepsilon} \left\{ \left(\frac{1}{|\gamma(u+w)-\gamma(u)|^2} - \frac{1}{w^2} \right) \langle \gamma'(u), \varphi'(u) \rangle \right.$$
$$\left. - 2 \left(\frac{\langle \gamma(u+w)-\gamma(u), \varphi(u+w)-\varphi(u) \rangle}{|\gamma(u+w)-\gamma(u)|^4} - \frac{\frac{\mathrm{d}}{\mathrm{d}\tau}\big|_{\tau=0} d_{\gamma+\tau\varphi}(u+w,u)}{|w|^3} \right) \right\} \mathrm{d}w\, \mathrm{d}u$$

where now

$$\tfrac{\mathrm{d}}{\mathrm{d}\tau}\big|_{\tau=0} d_{\gamma+\tau\varphi}(u+w,u) = |w| \int_0^1 \langle \gamma'(u+\theta w), \varphi'(u+\theta w) \rangle \, d\theta$$

for all $(u,w) \in \mathbb{R}/\mathbb{Z} \times (-\frac{1}{2}, \frac{1}{2})$. Hence,

$$\delta E^{(2)}(\gamma;\varphi) \overset{\varepsilon\searrow 0}{\longleftarrow} 2 \iint_{U_\varepsilon} \left\{ \left(\frac{1}{|\gamma(u+w)-\gamma(u)|^2} - \frac{1}{w^2} \right) \langle \gamma'(u), \varphi'(u) \rangle \right.$$
$$\left. - \left(\frac{\langle \gamma(u+w)-\gamma(u), \varphi(u+w)-\varphi(u) \rangle}{|\gamma(u+w)-\gamma(u)|^4} - \frac{\int_0^1 \langle \gamma'(u+\theta w), \varphi'(u+\theta w) \rangle \, d\theta}{w^2} \right) \right\} \mathrm{d}w\, \mathrm{d}u$$
$$= 2 \iint_{U_\varepsilon} \left\{ \left(\frac{1}{|\gamma(u+w)-\gamma(u)|^2} - \frac{1}{w^2} \right) \langle \gamma'(u), \varphi'(u) \rangle \right.$$
$$\left. - \left(\frac{\langle \gamma(u+w)-\gamma(u), \varphi(u+w)-\varphi(u) \rangle}{|\gamma(u+w)-\gamma(u)|^4} - \frac{\langle \gamma'(u), \varphi'(u) \rangle}{w^2} \right) \right\} \mathrm{d}w\, \mathrm{d}u$$
$$= 2 \iint_{U_\varepsilon} \left(\frac{\langle \gamma'(u), \varphi'(u) \rangle}{|\gamma(u+w)-\gamma(u)|^2} - \frac{\langle \gamma(u+w)-\gamma(u), \varphi(u+w)-\varphi(u) \rangle}{|\gamma(u+w)-\gamma(u)|^4} \right) \mathrm{d}w\, \mathrm{d}u.$$

□

3. Initial regularity: Proof of Theorem III

Note that if we consider the constant factors to be irrelevant with respect to the mathematical argument, for the sake of simplicity we will omit them in the calculations, writing $\lesssim, \gtrsim, \approx$ instead of $\leq, \geq$ and $=$.

Most techniques for dealing with critical partial differential equations of fractional order have been developed for equations on the whole Euclidean space. For that reason, we prefer working on the real line over working on the circle.

We will show that for every $u \in \mathbb{R}/\mathbb{Z}$ we have

$$|D|^{\frac{1}{2}} g' \in L^p((u - 1/20, u + 1/20))$$

for a $p > 2$. Due to the invariance of the problem under shifting the parametrization, it is enough to show this for $u = 1/2$, i.e.

$$|D|^{\frac{1}{2}} g' \in L^p((9/20, 11/20)). \tag{3.1}$$

Let $|D|^s \equiv (-\partial_x^2)^{s/2} \equiv (-\Delta)^{\frac{s}{2}}$ be the fractional Laplacian on $\mathbb{R}$. The inverse, $I_s \equiv (-\Delta)^{-\frac{s}{2}}$, $s \in (0, 1)$, is the Riesz potential by

$$I_s f(x) = c_s \int_{\mathbb{R}} \frac{f(y)}{|x - y|^{1-s}}\, dy. \tag{3.2}$$

In case of positive powers s of the Laplacian $|D|^s$, $s \in (0, 1)$, we use the corresponding formula

$$|D|^s f(x) = \tilde{c}_s \int_{\mathbb{R}} \frac{f(y) - f(x)}{|x - y|^{1+s}}\, dy. \tag{3.3}$$

For a detailed introduction to the fractional Laplacian we refer to, e.g., [DNPV11], [Sch10, Section 2.5].

To switch from the circle to the real line, we interpret functions on $\mathbb{R}/\mathbb{Z}$ as functions on $\mathbb{R}$ which are periodic with period 1. We then choose a cutoff function $\eta \in C_0^\infty(\mathbb{R})$, $\eta \equiv 1$ on $[-3, 4]$ and consider

$$g = \eta \cdot \gamma \tag{3.4}$$

instead of γ. We will show that $|D|^{\frac{1}{2}} g' \in L^p((17/40, 23/40), \mathbb{R}^n)$, for some $p > 2$ to conclude the proof.

Before we begin to outline the structure of the proof, let us shortly recapitulate the notion of Lorentz spaces and the main properties we are going to use in this article. For a measurable function $f : \Omega \to \mathbb{R}$ and $\Omega \subset \mathbb{R}$ one considers the *decreasing rearrangement*

$$f^*(t) := \inf \left\{ s > 0 : \mathcal{L}^1(\{|f| > s\}) \leq t \right\}$$

where $\mathcal{L}^1$ denotes the Lebesgue measure. We define

$$|f|_{(p,q),\Omega} := \begin{cases} \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t} \right)^{1/q} & \text{if } p, q \in [1, \infty), \\ \sup_{t>0} t^{1/p} f^*(t) & \text{if } q = \infty. \end{cases}$$

To prevent technical problems, unless $p \in (1, \infty)$ we will only take the spaces $L^{1,1} = L^1$ and $L^{\infty,\infty} = L^\infty$ into consideration.

Furthermore, $|f|_{(p,p)} \approx \|f\|_{L^p}$ for all $p \in [1,\infty]$. If $\Omega = \mathbb{R}$ we will omit Ω in the notation. Though $|\cdot|_{(p,q),\Omega}$ is not a norm, as it does not obey the triangle inequality, there is a norm $\|\cdot\|_{(p,q),\Omega}$ on the Lorentz spaces which is equivalent to $|\cdot|_{(p,q),\Omega}$. These norms satisfy a *Hölder inequality*, i.e., for $p_1, p_2, p \in [1,\infty)$ and $q_1, q_2, q \in [1,\infty]$ with $1/p_1 + 1/p_2 = 1/p$ and $1/q_1 + 1/q_2 = 1/q$ we have

$$\|fg\|_{(p,q),\Omega} \lesssim \|f\|_{(p_1,q_1),\Omega}\, \|g\|_{(p_2,q_2),\Omega}. \tag{3.5}$$

For $p_1, p_2, p \in (1,\infty)$ and $q_1, q_2 \in [1,\infty]$ with $1/p_1 + 1/p_2 = 1/p + 1$ and $1/q_1 + 1/q_2 = 1/q$ we have the *Young-O'Neil inequality* [Hun66]

$$\|f * g\|_{(p,q),\Omega} \lesssim \|f\|_{(p_1,q_1),\Omega}\, \|g\|_{(p_2,q_2),\Omega}. \tag{3.6}$$

Furthermore, we have the *Sobolev inequality*

$$\|I_s f\|_{(p^*,q)} \lesssim \|f\|_{(p,q)}$$

for all $s \geq 0$, $p \in [1,\infty)$, $q \in [1,\infty]$ and $p^* := \frac{p}{1-sp} \in [1,\infty)$. Further information and proofs can be found in [Hun66, Gra08, Tar07].

The main reason for using Lorentz spaces in the context of critical equations, i.e. equations to which standard Gagliardo-Nirenberg-Sobolev embeddings cannot be applied to gain regularity, is the following fact. Although for functions f the L^2-norm of $|D|^{\frac{1}{2}} f$ does not control the L^∞-norm of f, the $L^{2,1}$ norm does, i.e. we have the estimate

$$\|f\|_\infty \lesssim \||D|^{\frac{1}{2}} f\|_{(2,1)}.$$

We will also need this in the more general form

$$\|f\|_\infty \lesssim \||D|^s f\|_{(\frac{1}{s},1)} \quad \text{for all } s \in (0,1).$$

In order to prove our regularity result, we will prove a Dirichlet growth theorem for the weak $H^{1/2}$-energy of γ' on balls in a manner comparable to [DLR11a, Sch12] – which are as well in the setting of sphere-valued mappings. In contrast to these papers, the techniques from [DLR11b, DL11, Sch11] deal with a more general setting, but have to work (as we will here) with estimates of the $L^{2,\infty}$-norm instead of the L^2-norm. Note nevertheless that our right-hand side is very different from their's. In order to obtain the estimates of the norms $\||D|^{\frac{1}{2}} g'\|_{(2,\infty)}$ on small balls, we will have to use new arguments.

To prove the regularity theorem, we begin with an approach appearing in [DLR11a, Sch12] and divide $|D|^{\frac{1}{2}} g'$ into the part parallel to g' (and thus *normal* to the sphere $\mathbb{S}^{n-1}$) and the term perpendicular to g' (*tangential* to the sphere). More precisely, we use that for $p \in [1,\infty)$, $q \in [1,\infty]$

$$\||D|^{\frac{1}{2}} g'\|_{(p,q),B_r} \lesssim \|\langle g', |D|^{\frac{1}{2}} g'\rangle\|_{(p,q),B_r} + \sup_{\omega_{ij}} \|g'_j \omega_{ij} |D|^{\frac{1}{2}} g'_i\|_{(p,q),B_r}, \tag{3.7}$$

where the supremum is over $\omega_{ij} = -\omega_{ji} \in \{-1,0,1\}$. For a detailed version of this linear algebraic fact, the interested reader is referred to the appendix of [DLS12].

3.1. Estimate of the normal part

We have

$$\langle g', |D|^{\frac{1}{2}} g' \rangle = -\frac{1}{2} H_{\frac{1}{2}}(g', g') + \frac{1}{2} |D|^{\frac{1}{2}} |g'|^2, \tag{3.8}$$

where

$$H_s(a, b) := |D|^s(ab) - a\,|D|^s b - b\,|D|^s a. \tag{3.9}$$

Note that for any $s \in (0, 1)$, we have

$$|D|^s |g'|^2 \overset{(3.4)}{=} |D|^s \eta^2 + 2|D|^s(\eta' \gamma' g) + |D|^s(|\eta'|^2 |\gamma|^2) \in L^\infty((0, 1)). \tag{3.10}$$

In fact, $|D|^s \eta^2 \in L^\infty$ by interpolation inequalities. For the remaining terms we use the quasi-locality, Lemma A.1, and the support of η and η'.

As in [Sch11] we will use pointwise estimates for H_s and some quantitative version of the quasi-locality to estimate the normal part of $|D|^{\frac{1}{2}} g'$:

Lemma 3.1 (Normal part). *For any $s \in [0, \frac{1}{2})$ there exists $\theta > 0$ such that for any $B_r \subset [0, 1]$, $\Lambda > 4$*

$$\| |D|^s \langle g', |D|^{\frac{1}{2}} g' \rangle \|_{(\frac{2}{1+2s}, \infty), B_r} \lesssim \| |D|^{\frac{1}{2}} g' \|^2_{(2,\infty), B_{\Lambda r}} + \Lambda^{-\theta} \| |D|^{\frac{1}{2}} g' \|_{(2,\infty),\mathbb{R}} \sum_{k=1}^{\infty} 2^{-\theta k} \| |D|^{\frac{1}{2}} g' \|_{(2,\infty), B_{2^k \Lambda r}} + r^{\frac{1}{2}+s}. \tag{3.11}$$

For the readers' convenience, a proof will be given in the appendix.

3.2. Estimate of the tangential part

It then remains to estimate the part normal to g' (tangential to the sphere), i.e. for $\omega_{ij} = -\omega_{ji} \in \{-1, 0, 1\}$, $1 \le i, j \le n$ we need to estimate suitable norms on small balls of the term $g'_j \omega_{ij} |D|^{\frac{1}{2}} g'_i$.

We have

$$\begin{aligned} \int_{\mathbb{R}} g'_j \omega_{ij} |D|^{\frac{1}{2}} g'_i \; |D|^{\frac{1}{2}} \varphi &= \int_{\mathbb{R}} |D|^{\frac{1}{2}} g'_i \; |D|^{\frac{1}{2}} \big(\omega_{ij} g'_j \varphi\big) - \int_{\mathbb{R}} |D|^{\frac{1}{2}} g'_i \; \omega_{ij} |D|^{\frac{1}{2}} g'_j \; \varphi - \int_{\mathbb{R}} |D|^{\frac{1}{2}} g'_i \; \omega_{ij} H_{1/2}(g'_j, \varphi) \\ &= \int_{\mathbb{R}} |D|^{\frac{1}{2}} g'_i \; |D|^{\frac{1}{2}} \big(\omega_{ij} g'_j \varphi\big) - \int_{\mathbb{R}} |D|^{\frac{1}{2}} g'_i \; \omega_{ij} H_{1/2}(g'_j, \varphi) \end{aligned} \tag{3.12}$$

where we have used that due to $\omega_{ij} = -\omega_{ji}$ the second term on the right-hand side of the first line vanishes.

The second term can be estimated analogously to similar terms in [DLR11b, DL11, Sch11] using again quasi-locality together with Sobolev embeddings.

Lemma 3.2 (Tangential part). *There is $\theta > 0$, $s_0 \in (0, \frac{1}{2})$ such that for all $\phi \in C_0^\infty(B_r)$, $\Lambda > 16$, $s \in [0, s_0)$ we have*

$$\begin{aligned}&\int |D|^{\frac{1}{2}} g_i' \omega_{ij} H_{1/2}(g_j', \phi) dx \\ &\lesssim \||D|^{-s+1/2}\phi\|_{(\frac{2}{1-2s},1)} \left(\||D|^{\frac{1}{2}} g'\|^2_{(2,\infty),B_{\Lambda r}} + \Lambda^{-\theta} \||D|^{\frac{1}{2}} g'\|_{(2,\infty),\mathbb{R}} \sum_{k=1}^{\infty} 2^{-\theta(k-1)} \||D|^{\frac{1}{2}} g'\|_{(2,\infty),B_{2^k\Lambda r}} \right).\end{aligned}$$

A proof is provided in the appendix.

It remains to estimate the first term on the right-hand side of Equation (3.12) for which we will use the Euler-Langrange equation (1.6).

Combining this equation with the formula

$$\begin{aligned}&\int_{\mathbb{R}} |D|^{\frac{1}{2}} f_1' \, |D|^{\frac{1}{2}} f_2' \\ &\quad = c \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \left(\langle f_1'(u), f_2'(u) \rangle - \frac{\langle f_1(u+w) - f_1(u), f_2(u+w) - f_2(u) \rangle}{w^2} \right) \frac{dw}{w^2} du \end{aligned} \tag{3.13}$$

due to He [He99, Proposition 2] we get the following estimate which contains all the information of the Euler-Langrange equation we need to proceed in the proof:

Lemma 3.3 (Essential estimate of the Euler-Lagrange equation). *There is a constant $C < \infty$ such that*

$$\int_{\mathbb{R}} \left\langle |D|^{\frac{1}{2}} g', |D|^{\frac{1}{2}} \phi \right\rangle \leq C \int_{\mathbb{R}} |\phi(u)| \Gamma(u) du + C \|\phi\|_{L^2} \tag{3.14}$$

for any $\phi \in C_0^\infty((4/10, 6/10), \mathbb{R}^n)$ with $\langle \phi, \gamma' \rangle \equiv 0$ where

$$\Gamma(u) := \int_{(-1,1)^3} \int_{-1/4}^{1/4} \frac{|g'(u) - g'(u + s_2 w)| \, |g'(u + s_3 w) - g'(u + s_4 w)|^2}{|w|^2} \, dw \, ds.$$

The heart of the proof of Theorem III is the following pointwise estimate of the most problematic term $\Gamma(u)$, which permits to localize it and which afterwards will be transformed into a bound of its L^1-norm.

Lemma 3.4 (Estimate of the critical term). *We have*

$$\Gamma(u) \lesssim \left| |D|^{-\frac{1}{12}} |D|^{\frac{1}{2}} g'|(u) \right|^2 \, |D|^{-\frac{1}{3}} |D|^{\frac{1}{2}} g'|(u) + |D|^{-\frac{1}{12}} |D|^{\frac{1}{2}} g'|(u) \, |D|^{-\frac{1}{4}} \left| |D|^{-\frac{1}{12}} |D|^{\frac{1}{2}} g'|(u) \right|^2 \tag{3.15}$$

almost everywhere.

In fact, Lemmata 3.3 and 3.4 provide the essential new estimates in this article on which our entire reasoning relies crucially. Using these and the above-mentioned improved Sobolev embeddings for Lorentz spaces, we are ready to show the following estimate. This will allow us to prove a Dirichlet growth of the $L^{2,\infty}$-norm of $|D|^{\frac{1}{2}}g'$.

Lemma 3.5. *There exist $R > 0$, $s_0 \in (0, \frac{1}{2})$, and $\sigma > 0$ such that for any $\Lambda > 2$, $s \in [0, s_0)$, $B_{\Lambda r} \subset (\frac{5}{10}, \frac{6}{10})$, $r \in (0, \Lambda^{-1}R)$ and $\varphi \in C_0^\infty(B_r)$ we have*

$$\begin{aligned}&\int g_i'\omega_{ij}|D|^{\frac{1}{2}}g_j'\,|D|^{\frac{1}{2}}\varphi\\ &\lesssim \||I_s|D|^{\frac{1}{2}}\varphi\|_{(\frac{2}{1-2s},1)}\left(r^\sigma + \||D|^{\frac{1}{2}}g'\|^2_{(2,\infty),B_{\Lambda r}} + \Lambda^{-\theta}\||D|^{\frac{1}{2}}g'\|_{(2,\infty),\mathbb{R}}\sum_{k=1}^\infty 2^{-\theta k}\,\||D|^{\frac{1}{2}}g'\|_{(2,\infty),B_{2^k\Lambda r}}\right).\end{aligned}$$

3.3. Conclusion of the proof of Theorem III

Combining Lemma 3.5 with Proposition A.3, we get

$$\begin{aligned}&\||D|^s\left(g_i'\omega_{ij}|D|^{\frac{1}{2}}g_j'\right)\|_{(\frac{2}{2s+1},\infty),B_r}\\ &\lesssim \left(\left(\Lambda^2 r\right)^\sigma + \||D|^{\frac{1}{2}}g'\|^2_{(2,\infty),B_{\Lambda^3 r}} + \||D|^{\frac{1}{2}}g'\|_{(2,\infty),\mathbb{R}}\Lambda^{-\theta}\sum_{k=1}^\infty 2^{-\theta k}\,\||D|^{\frac{1}{2}}g'\|_{(2,\infty),B_{\Lambda^3 2^k r}}\right)\\ &\quad + \Lambda^{-\theta}r^{-s}\sum_{k=1}^\infty 2^{-\theta k}\,\||D|^{\frac{1}{2}}g'\|_{(\frac{2}{1+2s},\infty),B_{\Lambda 2^k r}}\\ &\lesssim \left(\cdots\right) + \Lambda^{s-\theta}\sum_{k=1}^\infty 2^{(s-\theta)k}\,\||D|^{\frac{1}{2}}g'\|_{(2,\infty),B_{\Lambda 2^k r}} \qquad (3.16)\end{aligned}$$

for some $\theta > 0$ uniformly in Λ.

Let us first use this for $s = 0$, to get in view of (3.7) and Lemma 3.1 that for all $\varepsilon > 0$ we have for sufficiently small $r > 0$, $B_r \subset \left(\frac{4}{10}, \frac{6}{10}\right)$ and big enough Λ

$$\||D|^{\frac{1}{2}}g'\|_{(2,\infty),B_r} \le C\left(\left(\Lambda^2 r\right)^\sigma + \varepsilon\,\||D|^{\frac{1}{2}}g'\|_{(2,\infty),B_{\Lambda^3 r}} + (1 + \||D|^{\frac{1}{2}}g'\|_{(2,\infty),\mathbb{R}})\Lambda^{-\theta}\sum_{k=2}^\infty 2^{-\theta k}\,\||D|^{\frac{1}{2}}g'\|_{(2,\infty),B_{2^k\Lambda^3 r}}\right)$$

uniformly in Λ and ε.

Let us fix such an r_0, and consider the above equation for $\Lambda = 2^{m/3}$, $r = 2^{-m-k}r_0$ (w.l.o.g. $r_0 = 1$). Setting $b_0 := \||D|^{\frac{1}{2}}g'\|_{(2,\infty)}$ and $b_k := \||D|^{\frac{1}{2}}g'\|_{(2,\infty),B_{2^{-k}}}$ the above estimate gives

$$b_{k+m} \le C2^{-\sigma(k+m/3)} + \varepsilon b_k + C2^{-\theta m/3}\left(\sum_{l=1}^k 2^{-\theta l}b_{k-l} + C2^{-\theta k}\right)$$

for every $k \in \mathbb{N}_0$ where $C < \infty$ does not depend on k.

Using the iteration argument, Lemma A.8, leads to

$$\||D|^{\frac{1}{2}} g'\|_{(2,\infty),B_r} \leq C_\gamma \, r^{\tilde{\sigma}} \tag{3.17}$$

for all $B_r \subset \left(\frac{9}{20}, \frac{11}{20}\right)$ and r small enough.

This immediately implies Hölder regularity of the solution g'. Instead, let us iterate the above argument to get $|D|^{\frac{1}{2}} g' \in L^p_{loc}((9/20, 11/20))$, for some $p > 2$.

Let us assume without loss of generality that $\tilde{\sigma} < \frac{\theta}{2}$. Then, plugging (3.17) into (3.16) one obtains

$$\||D|^s \left(g'_i \omega_{ij} |D|^{\frac{1}{2}} g'_j\right)\|_{(\frac{2}{2s+1},\infty),B_r} \lesssim C_{\Lambda,\gamma,s,R} \, r^{\min\{\sigma,\tilde{\sigma},\tilde{\sigma}-s,\theta/2\}}$$

for small enough $0 < s < (0, \tilde{\sigma})$, $r > 0$, and $B_r \subset [9/20, 11/20]$. Here we have used that $2^k \Lambda^3 r \geq R_0$ for some $R_0 > 0$ leads to $2^{-\theta k} \leq 2^{-\theta k/2} \left(\frac{\Lambda^3}{R_0}\right)^{\theta/2} r^{\theta/2}$. So the series in (3.16) converge and are bounded by some small positive power of r.

On the other hand, (3.17) and (3.11) together imply for any small enough $r > 0$, $\Lambda := r^{-\frac{1}{2}}$, and $B_r \subset \left(\frac{9}{20}, \frac{11}{20}\right)$

$$\begin{aligned} \||D|^{\frac{1}{4}} \langle g', |D|^{\frac{1}{2}} g' \rangle\|_{(\frac{4}{3},\infty),B_r} &\overset{(3.11)}{\lesssim} \||D|^{\frac{1}{2}} g'\|^2_{(2,\infty),B_{r^{\frac{1}{2}}}} + r^{\frac{\theta}{2}} \, \||D|^{\frac{1}{2}} g'\|^2_{(2,\infty),\mathbb{R}} + r^{\frac{3}{4}} \\ &\overset{(3.17)}{\leq} C_\gamma \left(r^{\tilde{\sigma}} + r^{\frac{\theta}{2}} + r^{\frac{3}{4}}\right). \end{aligned}$$

That is, $|D|^s \left(g'_i \omega_{ij} |D|^{\frac{1}{2}} g'_j\right)$, and $|D|^{\frac{1}{4}} \langle g', |D|^{\frac{1}{2}} g' \rangle$ both belong locally to a Morrey space $\mathcal{L}^{(p,q),\lambda}_{loc}$ on $\left(\frac{9}{20}, \frac{11}{20}\right)$. More precisely, for some $\lambda \in (0,1)$ we have

$$\begin{aligned} |D|^s \left(g'_i \omega_{ij} |D|^{\frac{1}{2}} g'_j\right) &\in \mathcal{L}^{(\frac{2}{2s+1},\infty),\lambda}_{loc}\left(\frac{9}{20}, \frac{11}{20}\right), \\ |D|^{\frac{1}{4}} \langle g', |D|^{\frac{1}{2}} g' \rangle &\in \mathcal{L}^{(\frac{4}{3},\infty),\lambda}_{loc}\left(\frac{9}{20}, \frac{11}{20}\right). \end{aligned}$$

The boundedness of Riesz potentials on Morrey spaces, as shown in [Ada75], implies that for some $p > 2$

$$g'_i \omega_{ij} |D|^{\frac{1}{2}} g'_j, \ \langle g', |D|^{\frac{1}{2}} g' \rangle \in L^p_{loc}\left(\frac{9}{20}, \frac{11}{20}\right).$$

Together, using (3.7), we have shown

$$|D|^{\frac{1}{2}} g'_j \in L^p_{loc}\left(\frac{9}{20}, \frac{11}{20}\right),$$

which finishes the proof of Theorem III. □

3.4. Proof of Lemma 3.3

The main idea is to use ϕ as the derivative of a test function for the Euler-Lagrange equation. Of course this is not possible generally, but with some precaution we can actually do it up to a benign error term: For $\phi \in C_0^\infty((4/10, 6/10))$ we set

$$h(u) := \int_0^u \phi(v)dv - au$$

where $a = \int_0^1 \phi(v)dv$ is chosen such that $h(0) = h(1)$. Hence, if we set

$$\begin{aligned} h_\pi(k+t) &:= h(t), \quad \forall t \in [0,1), k \in \mathbb{Z}, \\ \phi_\pi(k+t) &:= \phi(t), \quad \forall t \in [0,1), k \in \mathbb{Z}, \end{aligned}$$

h_π is a smooth one periodic function satisfying

$$h_\pi'(u) = \phi_\pi - a.$$

and, as we assume in Lemma 3.3 that $\langle \phi, \gamma' \rangle \equiv 0$, also

$$\langle \phi_\pi, \gamma' \rangle \equiv 0.$$

Since γ is a stationary point of the Möbius energy, testing the equation (1.6) with h_π, (recall (1.4), (1.5),(1.3))

$$Q(\gamma, h_\pi) := \lim_{\varepsilon \to 0} Q_\varepsilon(\gamma, h_\pi) = T_1(\gamma, h_\pi) + T_2(\gamma, h_\pi). \tag{3.18}$$

As ϕ_π is perpendicular to γ' we can estimate the term $T_1(\gamma, h_\pi)$ by

$$\begin{aligned} T_1(\gamma, h_\pi) &= \int_{-1}^{1} \int_{-1/2}^{1/2} \langle \gamma'(u), \phi_\pi(u) - a \rangle \left(\frac{1}{|\gamma(u+w) - \gamma(u)|^2} - \frac{1}{w^2} \right) \\ &\leq |a| \, E^{(2)}(\gamma) \leq C \|\phi\|_{L^1} \leq C \|\phi\|_{L^2}. \end{aligned} \tag{3.19}$$

As for the remaining terms $Q(\gamma, h_\pi)$ and $T_2(\gamma, h_\pi)$, we will identify them essentially with the left-hand side of (3.14) and the Γ-term on the right-hand side of (3.14), respectively. A technical detail one has to take into account here, is that the domain of (3.18) is the torus $\mathbb{R}/\mathbb{Z}$, whereas the respective domain in (3.14) is the real line $\mathbb{R}$. To estimate the other terms, let us introduce for $f_1, f_2 \in H^{3/2}_{loc}(\mathbb{R}, \mathbb{R}^n)$ the operators

$$\tilde{Q}_\varepsilon(f_1, f_2) := \int_0^1 \int_{I_\varepsilon} \left(\langle f_1'(u), \ f_2'(u) \rangle - \frac{\langle f_1(u+w) - f(u), f_2(u+w) - f_2(u) \rangle}{w^2} \right) \frac{dw}{w^2} \, du$$

$$= \int_0^1 \int_{\tilde{I}_\varepsilon} \int_0^1 \int_0^1 \frac{\left(\left\langle f_1'(u),\ f_2'(u)\right\rangle - \left\langle f_1'(u+s_1w), f_2'(u+s_2w)\right\rangle\right)}{w^2} ds_1 ds_2 dw\, du,$$

where $\tilde{I}_\varepsilon := [-1/4, 1/4] \setminus [-\varepsilon, \varepsilon]$,

$$\tilde{Q} := \lim_{\varepsilon \searrow 0} \tilde{Q}_\varepsilon,$$

and

$$\begin{aligned}
&\tilde{T}_2(f_1, f_2) \\
&:= \int_0^1 \int_{-1/4}^{1/4} \langle f_1(u+w) - f_1(u),\ f_2(u+w) - f_2(u)\rangle \left(\frac{1}{|\gamma(u+w) - \gamma(u)|^4} - \frac{1}{|w|^4}\right) dw\, du \\
&= \int_0^1 \int_{-1/4}^{1/4} \int_0^1 \int_0^1 \left\langle f_1'(u+s_1w),\ f_2'(u+s_2w)\right\rangle w^2 \left(\frac{1}{|\gamma(u+w) - \gamma(u)|^4} - \frac{1}{|w|^4}\right) ds_1\, ds_2\, dw\, du.
\end{aligned} \tag{3.20}$$

Recall that $[-\frac{1}{2}, \frac{1}{2}]\setminus(-\varepsilon, \varepsilon)$ is used for the definition of Q and $\tilde{I}_\varepsilon := [-1/4, 1/4] \setminus [-\varepsilon, \varepsilon]$ is used for $\tilde{Q}$, so the difference only contains the set where $|w| > \frac{1}{4}$, thus $|w|^{-2}$ is not singular. Quantitatively, this reads as

$$|\tilde{Q}(\gamma, h) - Q(\gamma, h)| + |\tilde{T}_2(\gamma, h) - T_2(\gamma, h)| \leq C\|\gamma'\|_{L^2}\|h'\|_{L^2} \leq C\|\gamma'\|_{L^2}\|\phi\|_{L^2}, \tag{3.21}$$

where we have used that γ is bi-Lipschitz in order to deal with T_2.

We now compute

$$\begin{aligned}
\tilde{Q}_\varepsilon(\gamma, h_\pi) &= \int_0^1 \int_{\tilde{I}_\varepsilon} \int_0^1 \int_0^1 \left(\langle \gamma'(u),\ h_\pi'(u)\rangle - \langle \gamma'(u+s_1w), h_\pi'(u+s_2w)\rangle\right) \mathrm{d}s_1\, \mathrm{d}s_2 \frac{dw}{w^2}\, du \\
&= \int_0^1 \int_{\tilde{I}_\varepsilon} \int_0^1 \int_0^1 \left(\langle \gamma'(u),\ \phi_\pi\rangle - \langle \gamma'(u+s_1w), \phi_\pi(u+s_2w)\rangle\right) \mathrm{d}s_1\, \mathrm{d}s_2 \frac{dw}{w^2}\, du \\
&\quad - \int_0^1 \int_{\tilde{I}_\varepsilon} \int_0^1 \int_0^1 \left(\langle \gamma'(u),\ a\rangle - \langle \gamma'(u+s_1w), a\rangle\right) \mathrm{d}s_1\, \mathrm{d}s_2 \frac{dw}{w^2}\, du \qquad (3.22) \\
&= \int_0^1 \int_{\tilde{I}_\varepsilon} \int_0^1 \int_0^1 \left(\langle g'(u),\ \phi(u)\rangle - \langle g'(u+s_1w), \phi(u+s_2w)\rangle\right) \mathrm{d}s_1\, \mathrm{d}s_2 \frac{dw}{w^2}\, du. \qquad (3.23)
\end{aligned}$$

Here we have used that $\int_0^1 \gamma'(u+\tilde{w})du = 0$ for all $\tilde{w}$, i.e., the term (3.22) is constantly zero for any w. The term (3.23) essentially is the L^2-pairing of $|D|^{\frac{1}{2}}g'$ and $|D|^{\frac{1}{2}}\phi$: More precisely we will show

$$|\tilde{Q}(\gamma, h_\pi) - \int_{\mathbb{R}} |D|^{\frac{1}{2}} g' |D|^{\frac{1}{2}} \phi| \leq C_\gamma \|\phi\|_{L^2}. \tag{3.24}$$

Proof of (3.24). Using (3.13), we have for any $f_1, f_2 \in C_0^\infty(\mathbb{R})$ as $\varepsilon \to 0$

$$\begin{aligned} &\int_{\mathbb{R}} |D|^{\frac{1}{2}} f_1' |D|^{\frac{1}{2}} f_2' du + o(1) \\ &= c \int_{\mathbb{R}} \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \int_0^1 \int_0^1 \left(\langle f_1'(u), f_2'(u) \rangle - \langle f_1'(u+s_1 w), f_2'(u+s_2 w) \rangle \right) \mathrm{d}s_1 \, \mathrm{d}s_2 \frac{dw}{|w|^2} du. \end{aligned} \tag{3.25}$$

We now connect and (3.25) and (3.23). The technical problem is, that the integral of ϕ is not a feasible test-function for (3.25). Let therefore for $\eta_{[-10,10]} \in C_0^\infty((-11,11))$ and $\eta_{[-10,10]} \equiv 1$ in $[-10,10]$,

$$\psi := \left(u \mapsto \eta_{[-10,10]} \int_0^u \phi \right) \in C_0^\infty(\mathbb{R}).$$

Thus, ψ is a feasible test-function for (3.25), which ϕ is not. Moreover,

$$\psi' = \eta_{[-10,10]} \phi + \eta'_{[-10,10]} \int_0^u \phi \overset{\text{supp}\,\phi}{\equiv} \phi + \eta'_{[-10,10]} \int_0^u \phi. \tag{3.26}$$

We thus arrive at

$$\begin{aligned} \tilde{Q}_\varepsilon(\gamma, h) &\overset{(3.23)}{=} \int_0^1 \int_{\tilde{I}_\varepsilon} \int_0^1 \int_0^1 \left(\langle g'(u),\ \phi(u) \rangle - \langle g'(u+s_1 w), \phi(u+s_2 w) \rangle \right) \mathrm{d}s_1 \, \mathrm{d}s_2 \frac{dw}{w^2} \, du \\ &\overset{\text{supp}\,\phi,\, |w|,(3.26)}{=} \int_{\mathbb{R}} \int_{\tilde{I}_\varepsilon} \int_0^1 \int_0^1 \left(\langle g'(u),\ \psi'(u) \rangle - \langle g'(u+s_1 w), \psi'(u+s_2 w) \rangle \right) \mathrm{d}s_1 \, \mathrm{d}s_2 \frac{dw}{w^2} \, du \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \int_0^1 \int_0^1 \left(\langle g'(u),\ \psi'(u) \rangle - \langle g'(u+s_1 w), \psi'(u+s_2 w) \rangle \right) \mathrm{d}s_1 \, \mathrm{d}s_2 \frac{dw}{w^2} \, du \\ &\quad - \int_{\mathbb{R}} \int_{|w|>\frac{1}{4}} \int_0^1 \int_0^1 \left(\langle g'(u),\ \psi'(u) \rangle - \langle g'(u+s_1 w), \psi'(u+s_2 w) \rangle \right) \mathrm{d}s_1 \, \mathrm{d}s_2 \frac{dw}{w^2} \, du \end{aligned}$$

$$
\begin{aligned}
&\overset{(3.25)}{=} \int \langle |D|^{\frac{1}{2}} g', |D|^{\frac{1}{2}} \psi' \rangle \\
&\quad - \int_{\mathbb{R}} \int_{|w|>\frac{1}{4}} \int_0^1 \int_0^1 \left(\langle g'(u),\ \psi'(u) \rangle - \langle g'(u+s_1 w), \psi'(u+s_2 w) \rangle \right) \mathrm{d}s_1 \, \mathrm{d}s_2 \frac{dw}{w^2} \, du \\
&\quad + o(1) \\
&\overset{(3.26)}{=} \int \langle |D|^{\frac{1}{2}} g', |D|^{\frac{1}{2}} \phi \rangle + \int \langle |D|^{\frac{1}{2}} g', |D|^{\frac{1}{2}} (\eta'_{[-10,10]} \int_0^{\cdot} \phi) \rangle \\
&\quad - \int_{\mathbb{R}} \int_{|w|>\frac{1}{4}} \int_0^1 \int_0^1 \left(\langle g'(u),\ \psi'(u) \rangle - \langle g'(u+s_1 w), \psi'(u+s_2 w) \rangle \right) \mathrm{d}s_1 \, \mathrm{d}s_2 \frac{dw}{w^2} \, du \\
&\quad + o(1).
\end{aligned}
$$

Now, by usual interpolation and/or imbedding of Sobolev spaces, see, e.g. [Tar07, Sch10],

$$
\| |D|^{\frac{1}{2}} f \|_{2,\mathbb{R}} \lesssim \|f\|_{2,\mathbb{R}} + \|f'\|_{2,\mathbb{R}},
$$

we have

$$
\| |D|^{\frac{1}{2}} (\eta'_{[-10,10]} \int_0^{\cdot} \phi) \|_{2,\mathbb{R}} \lesssim \|\phi\|_1 + \|\phi\|_2 \overset{\text{supp}\,\phi}{\lesssim} \|\phi\|_2.
$$

Moreover,

$$
\begin{aligned}
&\int_{\mathbb{R}} \int_{|w|>\frac{1}{4}} \int_0^1 \int_0^1 \left(\langle g'(u),\ \psi'(u) \rangle - \langle g'(u+s_1 w), \psi'(u+s_2 w) \rangle \right) \mathrm{d}s_1 \, \mathrm{d}s_2 \frac{dw}{w^2} \, du \\
\lesssim\ & \int_{|w|>\frac{1}{4}} |w|^{-2} \, \|g'\|_2 \|\psi'\|_2 \\
\overset{(3.26)}{\lesssim}\ & \|g'\|_{2,\mathbb{R}} \Big(\|\phi\|_{1,\mathbb{R}} + \|\phi\|_{2,\mathbb{R}} \Big) \overset{\text{supp}\, g}{\lesssim} C_\gamma \, \|\phi\|_{2,\mathbb{R}}.
\end{aligned}
$$

Thus, we have shown that (3.24) holds. □

To estimate $\tilde{T}_2(\gamma, h)$ we calculate

$$
\tilde{T}_2(\gamma, h) = \int_0^1 \int_{-1/4}^{1/4} \int_0^1 \int_0^1 \langle \gamma'(u + s_1 w),\ \phi_\pi(u + s_2 w) + a \rangle \, w^2 \cdot
$$

$$\cdot\left(\frac{1}{|\gamma(u+w)-\gamma(u)|^4}-\frac{1}{|w|^4}\right)ds_1\,ds_2\,dw\,du$$

$$=\int_0^1\int_{-1/4}^{1/4}\int_0^1\int_0^1\langle\gamma'(u+s_1w),\ \phi(u+s_2w)-\phi(u+s_1w)+a\rangle\,w^2\cdot$$

$$\left(\frac{1}{|\gamma(u+w)-\gamma(u)|^4}-\frac{1}{|w|^4}\right)ds_1\,ds_2\,dw\,du$$

and using

$$w^2\left(\frac{1}{|\gamma(u+w)-\gamma(u)|^4}-\frac{1}{|w|^4}\right)=\frac{w^4}{|\gamma(u+w)-\gamma(u)|^4}\left(\frac{1-\frac{|\gamma(u+w)-\gamma(u)|^4}{w^4}}{w^2}\right)$$

$$\leq C\frac{1-\frac{|\gamma(u+w)-\gamma(u)|^2}{w^2}}{w^2}$$

$$=\frac{C}{2}\int_0^1\int_0^1\frac{|\gamma'(u+s_3w)-\gamma'(u+s_4w)|^2}{w^2}ds_3\,ds_4$$

we get

$$|\tilde{T}_2(\gamma,h)|\leq|a|E^{(2)}(\gamma)$$

$$+\int_0^1\int_{-1/4}^{1/4}\int_{(0,1)^4}\frac{|\gamma'(u+s_1w)-\gamma'(u+s_2w)|\,|\gamma'(u+s_3w)-\gamma'(u+s_4w)|^2\,|\phi(u+s_2w)|}{w^2}dsdw\,du$$

$$\leq C\|\phi\|_{L^2}+\int_0^1\int_{-1/4}^{1/4}\int_{(-1,1)^3}\frac{|\gamma'(u+s_1w)-\gamma'(u)|\,|\gamma'(u+s_3w)-\gamma'(u+s_4w)|^2\,|\phi(u)|}{w^2}dsdw\,du. \tag{3.27}$$

From (3.21),(3.19),(3.24), and (3.27) one gets the claim, since $\gamma'=g'$ on $[-1/4,5/4]$.

3.5. Proof of Lemma 3.4

Let

$$F(u):=\||D|^{\frac{1}{2}}g'(u)|.$$

Since $g',|D|^{\frac{1}{2}}g'\in L^2$, we obtain

$$g'(x)-g'(y)=I_{\frac{1}{2}}(|D|^{\frac{1}{2}}g')(x)-I_{\frac{1}{2}}(|D|^{\frac{1}{2}}g'(y))$$

$$= c_{\frac{1}{2}} \left(\int |x-\xi|^{-1+\frac{1}{2}} \, |D|^{\frac{1}{2}} g'(\xi) \, d\xi - \int |y-\xi|^{-1+\frac{1}{2}} \, |D|^{\frac{1}{2}} g'(\xi) \, d\xi \right),$$

then

$$|g'(x) - g'(y)| \lesssim \int_{\mathbb{R}} \| \xi - x|^{-1+\frac{1}{2}} - |\xi - y|^{-1+\frac{1}{2}} | \, F(\xi) \, d\xi,$$

and hence

$$\Gamma(u) \lesssim \int_{(-1,1)^3} \int_{\mathbb{R}^3} \int_{-1/4}^{1/4} F(\xi_1) \, F(\xi_2) \, F(\xi_3) \, k(\xi, u, s, w) \, dw \, d\xi \, ds,$$

where for almost every $s = (s_1, s_2, s_3) \in (-1,1)^3$, $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, $w \in \mathbb{R}$, $u \in \mathbb{R}$,

$$k(\xi, u, s, w) = \frac{m(\xi_1 - u, 0, s_1, w) \, m(\xi_2 - u, s_2, s_3, w) \, m(\xi_3 - u, s_2, s_3, w)}{|w|^2} \tag{3.28}$$

and

$$m(a, s, t, w) := \| a + sw|^{-\frac{1}{2}} - |a + tw|^{-\frac{1}{2}}|.$$

The characteristic behavior of k is as follows: The factors $m(\cdot, \cdot, \cdot, w)$ will behave like $|w|^{\delta}$ in a neighborhood of $w = 0$ such that they somewhat absorb the singular behaviour of $|w|^{-2}$, that is, k becomes integrable around $w = 0$. This is an effect very similar to the behaviour of $H_s(\cdot, \cdot)$, see Lemma A.5, as developed in [Sch11].

More precisely, we will derive the estimate

$$\int_{[-1,1]^3} k(\xi, u, s, w) \, \mathrm{d}s$$

$$\lesssim |w|^{3\delta-2} |\xi_1 - u|^{-1/2-\delta} \sum_{(\sigma(2),s(3)) \in \{(2,3),(3,2)\}} |\xi_{\sigma(2)} - u|^{-1/2-\delta} \int_{-1}^{1} |\xi_{\sigma(3)} - u + tw|^{-1/2-\delta} \, \mathrm{d}t \tag{3.29a}$$

$$+ |w|^{3\delta-2} |\xi_1 - u|^{-1/2-\delta} \int_{-1}^{1} |\xi_2 - u + tw|^{-1/2-\delta} |\xi_3 - u + tw|^{-1/2-\delta} \, \mathrm{d}t. \tag{3.29b}$$

We start with some abstract treatment of m.

In **case 1**,
$$\max(|a + sw|, |a + tw|) \geq 2|s - t||w|, \tag{3.30}$$

we obtain

$$|a + sw| \approx |a + tw|. \tag{3.31}$$

Applying the mean value theorem, for any $\delta \in (0, 1)$ we arrive at

$$m(a, s, t, w) \lesssim \max\left(|a + sw|^{-3/2}, |a + tw|^{-3/2} \right) |s - t||w|$$

$$
\begin{aligned}
&\lesssim \max\left(|a+sw|^{-1/2-\delta}, |a+tw|^{-1/2-\delta}\right) \max\left(|a+sw|^{-1+\delta}, |a+tw|^{-1+\delta}\right) |s-t||w| \\
&\overset{(3.30)}{\lesssim} \max\left(|a+sw|^{-1/2-\delta}, |a+tw|^{-1/2-\delta}\right) \underbrace{|s-t|^{\delta}}_{\leq 2^{\delta}} |w|^{\delta} \\
&\overset{(3.31)}{\lesssim} \min\left(|a+sw|^{-1/2-\delta}, |a+tw|^{-1/2-\delta}\right) |w|^{\delta}. \qquad (3.32)
\end{aligned}
$$

In **case 2**,
$$
\max\left(|a+sw|, |a+tw|\right) \leq 2|s-t||w|, \qquad (3.33)
$$
we immediately obtain
$$
\begin{aligned}
m(a,s,t,w) &\lesssim \max\left(|a+sw|^{-1/2}, |a+tw|^{-1/2}\right) \\
&\lesssim \max\left(|a+sw|^{-1/2-\delta}, |a+tw|^{-1/2-\delta}\right) \underbrace{|s-t|^{\delta}}_{\leq 2^{\delta}} |w|^{\delta}. \qquad (3.34)
\end{aligned}
$$

We begin with the first factor in (3.28). In case 1 we always have
$$
m(\xi_1 - u, 0, s_1, w) \lesssim |\xi_1 - u|^{-1/2-\delta} |w|^{\delta}. \qquad (3.35)
$$
In case 2 we have either $|\xi_1 - u + s_1 w| \geq \frac{1}{2}|\xi_1 - u|$ which immediately results in (3.35) or the opposite
$$
|\xi_1 - u + s_1 w| \leq \tfrac{1}{2}|\xi_1 - u| \qquad (3.36)
$$
which leads to
$$
\begin{aligned}
\int_{-1}^{1} \chi_{|\xi_1-u+s_1w|\leq|\xi_1-u|} m(\xi_1-u,0,s_1,w)\,\mathrm{d}s_1 &\lesssim |w|^{\delta} \int_{|\xi_1-u+s_1w|\leq|\xi_1-u|} |\xi_1-u+s_1w|^{-1/2-\delta}\,\mathrm{d}s_1 \\
&\lesssim |w|^{\delta-1} \int_{|\sigma|\leq|\xi_1-u|} |\sigma|^{-1/2-\delta}\,\mathrm{d}\sigma \lesssim |w|^{\delta-1}|\xi_1-u|^{1/2-\delta} \\
&\overset{(3.38)}{\lesssim} |\xi_1-u|^{-1/2-\delta}|w|^{\delta}, \qquad \text{for } \delta \in (0, \tfrac{1}{2}). \qquad (3.37)
\end{aligned}
$$

Here we made use of the fact that, given case 2 for $a := \xi_1 - u$ and (3.36),
$$
\begin{aligned}
|a| &\leq \min\left(|a+sw| + |s||w|, |a+tw| + |t||w|\right) \leq \min\left(|a+sw|, |a+tw|\right) + |w| \\
&\overset{(3.36)}{\leq} \tfrac{1}{2}|a| + |w|,
\end{aligned}
$$
implies
$$
|a| \leq 2|w|. \qquad (3.38)
$$

Applying (3.32) and (3.34), we arrive at

$$\begin{aligned}
&\int_{[-1,1]^3} k(\xi,u,s,w)\,\mathrm{d}s \\
&\lesssim |w|^{\delta-2}|\xi_1-u|^{-1/2-\delta}\iint_{[-1,1]^2} m(\xi_2-u,s_2,s_3,w)m(\xi_3-u,s_2,s_3,w)\,\mathrm{d}s_2\,\mathrm{d}s_3 \\
&\lesssim |w|^{3\delta-2}|\xi_1-u|^{-1/2-\delta}\iint_{[-1,1]^2} \mu_2\left(|\xi_2-u+s_2w|^{-1/2-\delta},|\xi_2-u+s_3w|^{-1/2-\delta}\right)\cdot \\
&\qquad\qquad \cdot\,\mu_3\left(|\xi_3-u+s_2w|^{-1/2-\delta},|\xi_3-u+s_3w|^{-1/2-\delta}\right)\mathrm{d}s_2\,\mathrm{d}s_3
\end{aligned} \tag{3.39}$$

where

$$\mu_i=\mu_i(\xi_i-u,s_2,s_3,w)\in\{\min,\max\},\qquad i=2,3,$$

depending on the respective case. If case 1 holds for at least one of the two factors in the integrand, say the first one, we may choose the argument of

$$\mu_2\left(|\xi_2-u+s_2w|^{-1/2-\delta},|\xi_2-u+s_3w|^{-1/2-\delta}\right)$$

which contains the same integration variable as the second one. This results in terms of type (3.29b). If, however, case 2 applies to both factors, the integral in (3.39) is bounded by

$$\begin{aligned}
&\iint_{[-1,1]^2}\left(|\xi_2-u+s_2w|^{-1/2-\delta}+|\xi_2-u+s_3w|^{-1/2-\delta}\right)\cdot \\
&\qquad\cdot\left(|\xi_3-u+s_2w|^{-1/2-\delta}+|\xi_3-u+s_3w|^{-1/2-\delta}\right)\mathrm{d}s_2\,\mathrm{d}s_3.
\end{aligned}$$

Expanding the integrand, the terms $|\xi_2-u+s_iw|^{-1/2-\delta}|\xi_3-u+s_iw|^{-1/2-\delta}$, $i=2,3$, lead us to (3.29b). For the two remaining terms we may separate the integrals which gives

$$\sum_{(\sigma(2),s(3))\in\{(2,3),(3,2)\}}\int_{-1}^{1}|\xi_{\sigma(2)}-u+s_2w|^{-1/2-\delta}\,\mathrm{d}s_2\int_{-1}^{1}|\xi_{\sigma(3)}-u+s_3w|^{-1/2-\delta}\,\mathrm{d}s_3.$$

One integral is kept in order to arrive at (3.29a), the other one is treated analogously to (3.36) and (3.37). In order to estimate $\Gamma(u)$, we obtain thus for $\delta_1,\delta_2\in(0,\frac{1}{2})$

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{(-1,1)^3} F(\xi_1)\,F(\xi_2)\,F(\xi_3)\,k(\xi,u,s,w)\,\mathrm{d}s\,\mathrm{d}\xi \\
\approx\;&\int_{-1}^{1}|w|^{-1+(3\delta-1)}|I_{\frac{1-2\delta}{2}}F(u)|^2\;I_{\frac{1-2\delta}{2}}F(u-tw)\,\mathrm{d}t \\
&+\int_{-1}^{1}|w|^{-1+(3\delta-1)}I_{\frac{1-2\delta}{2}}F(u)\;|I_{\frac{1-2\delta}{2}}F(u-tw)|^2\,\mathrm{d}t,
\end{aligned}$$

which implies

$$\begin{aligned}
\Gamma(u) \quad &\lesssim \quad \int_{-1}^{1}\int_{\tilde w} t^{2-3\delta}\,|\tilde w|^{-1+(3\delta-1)}|I_{\frac{1-2\delta}{2}}F(u)|^2\; I_{\frac{1-2\delta}{2}}F(u-\tilde w)\; t^{-1}\,\mathrm{d}\tilde w\,\mathrm{d}t \\
&\quad + \int_{-1}^{1}\int_{\tilde w} t^{2-3\delta}\,|\tilde w|^{-1+(3\delta-1)}I_{\frac{1-2\delta}{2}}F(u)\;|I_{\frac{1-2\delta}{2}}F(u-\tilde w)|^2\; t^{-1}\,\mathrm{d}\tilde w\,\mathrm{d}t \\
&\overset{\delta\in(\frac13,\frac12)}{\approx} \quad \int_{w}|w|^{-1+(3\delta-1)}|I_{\frac{1-2\delta}{2}}F(u)|^2\; I_{\frac{1-2\delta}{2}}F(u-w)\,\mathrm{d}w \\
&\quad + \int_{w}|w|^{-1+(3\delta-1)}I_{\frac{1-2\delta}{2}}F(u)\;|I_{\frac{1-2\delta}{2}}F(u-w)|^2\,\mathrm{d}w \\
&\approx \quad |I_{\frac{1-2\delta}{2}}F(u)|^2\; I_{\frac{4\delta-1}{2}}F(u) + I_{\frac{1-2\delta}{2}}F(u)\; I_{3\delta-1}|I_{\frac{1-2\delta}{2}}F(u)|^2.
\end{aligned}$$

Setting $\delta := \frac{5}{12}$, this is (3.15).

3.6. Proof of Lemma 3.5

Plugging together (3.12), Lemma 3.2, and Lemma 3.3 we get for small s and some $\theta > 0$ that

$$\begin{aligned}
&\int g_i'\omega_{ij}\,|D|^{\frac12}g_j'\,|D|^{\frac12}\varphi \\
&\lesssim \|\varphi\|_\infty\,\|\Gamma\|_{1,B_r} + \||I_s|D|^{\frac12}\varphi\|_{(\frac{2}{1-2s},1)}\cdot \\
&\quad\cdot\left(r^{\frac12} + \||D|^{\frac12}g'\|^2_{(2,\infty),B_{\Lambda r}} + \left\||D|^{\frac12}g'\right\|_{(2,\infty),\mathbb{R}}\sum_{k=2}^{\infty}\Lambda^{-\theta(k-1)}\||D|^{\frac12}g'\|_{(2,\infty),B_{\Lambda^k r}}\right).
\end{aligned}$$

By the estimate of Γ from Lemma 3.4, for $F := \||D|^{\frac12}g'|$,

$$\begin{aligned}
\|\Gamma\|_{1,B_r} \quad &\lesssim \quad \left\||I_{\frac{1}{12}}|F|\|^2\; I_{\frac13}|F| + I_{\frac{1}{12}}|F|\; I_{\frac14}\left(|I_{\frac{1}{12}}|F\|^2\right)\right\|_{1,B_r} \\
&\lesssim \quad \Theta\left(\left\|I_{\frac13}|F|\right\|_{(6,\infty),B_r} + \left\|I_{\frac{1}{12}}|F|\right\|_{(\frac{12}{5},\infty),B_r}\right),
\end{aligned}$$

where

$$\Theta := \left\|I_{\frac{1}{12}}|F|\right\|^2_{(\frac{12}{5},2),B_r} + \left\|I_{\frac14}\left(|I_{\frac{1}{12}}|F\|^2\right)\right\|_{(\frac{12}{7},1),B_r}.$$

Observe,

$$\begin{aligned}
\|I_{\frac13}F\|_{(6,\infty)} \quad &\lesssim \quad \|F\|_{(2,\infty)} \\
\|I_{\frac{1}{12}}F\|_{(\frac{12}{5},2)} \quad &\lesssim \quad \|F\|_2,
\end{aligned}$$

and because $\frac{5}{12} + \frac{5}{12} + \frac{1}{6} = 1$,

$$\begin{aligned} \|I_{\frac{1}{12}} F\|_{(\frac{12}{5},q)} &\lesssim \|F\|_{(2,q)}, \\ \|I_{\frac{1}{4}} |I_{\frac{1}{12}} F|^2\|_{(\frac{12}{7},1)} &\lesssim \|\|I_{\frac{1}{12}} F|^2\|_{(\frac{6}{5},1)} = \|\|I_{\frac{1}{12}} F\|\|^2_{(\frac{12}{5},2)} \lesssim \|F\|_2. \end{aligned}$$

Consequently, Θ is uniformly small, if r is small enough. In order to conclude, it only remains to apply Lemma A.2 to $f := |F|$. □

4. Bootstrapping: Proof of Theorem IV

In Theorem III we have shown, that $|D|^{\frac{1}{2}} g' \in L^p$ for some $p > 2$. We now work with Bessel-potential / Sobolev spaces $H^{s,q}$, cf. [RS96, Tar07, Tri83], and the fact that $|D|^{\frac{1}{2}} g' \in L^p$ readily implies that $g' \in H^{\frac{3}{2},\tilde{p}}$ for some $\tilde{p} \in (2, p)$. The proof of Theorem IV relies on the decomposition of the first variation

$$\delta E^{(2)}(\gamma, h) = 2 \lim_{\varepsilon \searrow 0} \iint_{U_\varepsilon} \left(\frac{\langle \gamma'(u), h'(u) \rangle}{|\gamma(u+w) - \gamma(u)|^2} - \frac{\langle \gamma(u+w) - \gamma(u), h(u+w) - h(u) \rangle}{|\gamma(u+w) - \gamma(u)|^4} \right) \mathrm{d}w \, \mathrm{d}u,$$

U_ε as in (1.2), into

$$2 \left(Q(\gamma, h) - T_1(\gamma, h) - T_2(\gamma, h) \right).$$

For a stationary point of the Möbius energy we have

$$Q(\gamma, h) = T(\gamma, h) := T_1(\gamma, h) + T_2(\gamma, h)$$

for all $h \in H^{3/2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$.

Let us bring these terms in a common form. Using

$$\begin{aligned} \frac{1}{|\gamma(u+w) - \gamma(u)|^\alpha} - \frac{1}{|w|^\alpha} &= \frac{|w|^\alpha}{|\gamma(u+w) - \gamma(u)|^\alpha} \left(\frac{1 - \frac{|\gamma(u+w) - \gamma(u)|^\alpha}{|w|^\alpha}}{|w|^\alpha} \right) \\ &= G^\alpha \left(\frac{\gamma(u+w) - \gamma(u)}{w} \right) \left(\frac{2 - 2\frac{|\gamma(u+w) - \gamma(u)|^2}{|w|^2}}{|w|^\alpha} \right) \\ &= \int_0^1 \int_0^1 G^\alpha \left(\frac{\gamma(u+w) - \gamma(u)}{w} \right) \left(\frac{|\gamma'(u+\tau_1 w) - \gamma'(u+\tau_2 w)|^2}{|w|^\alpha} \right) d\tau_1 d\tau_2 \end{aligned}$$

where

$$G^\alpha(z) := \frac{1}{2|z|^\alpha} \cdot \frac{1 - |z|^\alpha}{1 - |z|^2}$$

is an analytic function away from the origin for $\alpha \geq 2$. We hence get

$$T_1(\gamma, h) = -\int_0^1 \int_0^1 T^2_{0,0,\tau_1,\tau_2}(h) d\tau_1 d\tau_2$$

$$T_2(\gamma, h) = \int_0^1 \int_0^1 \int_0^1 \int_0^1 T^4_{s_1,s_2,\tau_1,\tau_2}(h) d\tau_1 d\tau_2 ds_1 ds_2$$

where

$$T^\alpha_{s_1,s_2,\tau_1,\tau_2}(h) := \int_0^1 \int_{-1/2}^{1/2} G^\alpha \left(\frac{\gamma(u+w) - \gamma(u)}{w} \right) \frac{|\gamma'(u+\tau_1 w) - \gamma'(u+\tau_2 w)|^2}{w^2} \gamma'(u+s_1 w) h'(u+s_2 w) dw\, du.$$

In the rest of this section, we will derive some estimates for the linear operators $T^\alpha_{s_1,s_2,\tau_1,\tau_2}$ that do not depend on s_1, s_2, τ_1, and τ_2.

For this task, we will work with the Besov spaces $B^s_{p,q}$. Given $s \in (0,1)$ and $p, q \in [1, \infty)$ one way to define the norm on these spaces is to set

$$|f|_{B^s_{p,q}} := \left(\int_{-1/2}^{1/2} \frac{\left(\int_{\mathbb{R}/\mathbb{Z}} |f(u+w) - f(u)|^p du \right)^{q/p}}{|w|^{1+sq}} dw \right)^{1/q}$$

and to put

$$\|f\|_{B^s_{p,q}} := \|f\|_{L^p} + |f|_{B^s_{p,q}}.$$

The Besov space $B^s_{p,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ then consists of all functions $f \in L^p$ with $|f|_{B^s_{p,q}} < \infty$.

Apart from this definition we just need the Sobolev embedding

$$H^{\tilde{s},\tilde{p}} \subset B^s_{p,q}$$

if $s < \tilde{s}$ and $s - \frac{1}{p} < \tilde{s} - \frac{1}{\tilde{p}}$ which can be found in textbooks like, e.g., [Tri83, RS96, Tar07]. We also refer to [DNPV11].

The proof relies furthermore on the following rules for Bessel potential spaces.

Lemma 4.1 (Fractional Leibniz rule [CM78], [RS96, Lem. 5.3.7/1 (i)]). *Let $f \in H^{s,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, $g \in H^{s,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ with $s > 0$, $p, q, r \in (1, \infty)$ and $1/p + 1/q = 1/r$.*

Then $fg \in H^{s,r}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ and

$$\|fg\|_{H^{s,r}} \leq C \left(\|f\|_{H^{s,p}} \|g\|_{L^q} + \|f\|_{L^p} \|g\|_{H^{s,q}} \right).$$

For the following statement, one mainly has to treat $\left\|(D^k\psi)\circ f\right\|_{H^{\sigma,p}}$ for $k \in \mathbb{N}\cup\{0\}$ and $\sigma \in (0,1)$ which is e. g. covered by [RS96, Thm. 5.3.6/1 (i)].

Lemma 4.2 (Fractional chain rule). *Let $f \in H^{s,p}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$, $s>0$, $p\in(1,\infty)$. If $\psi\in C^\infty(\mathbb{R})$ is globally Lipschitz continuous and ψ and all its derivatives vanish at 0 then $\psi\circ f\in H^{s,p}$ and*

$$\|\psi\circ f\|_{H^{s,p}} \le C\|\psi\|_{C^k}\|f\|_{H^{s,p}}$$

where k is the smallest integer greater than or equal to s.

The key to the proof of Theorem IV is the following lemma.

Lemma 4.3. *Let $\gamma\in H^{\frac{3}{2}+\beta_0,q}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$, $\alpha\ge 2$, and $\beta_0\ge\beta\ge 0$, $p,q\in(1,\infty)$ be such that $\beta_0-1/q>\beta-\frac{1}{2p}$. Then for all $\tau_1,\tau_2,s_1\in[0,1]$ the function*

$$g(u) := \int_{-1/2}^{1/2} G^\alpha\left(\frac{\gamma(u+w)-\gamma(u)}{w}\right)\frac{|\gamma'(u+\tau_1 w)-\gamma'(u+\tau_2 w)|^2}{w^2}\gamma'(u+s_1 w)dw$$

is in $H^{\beta,p}$. Furthermore, there is a constant $C<\infty$ depending on $\|\gamma\|_{H^{3/2+\beta_0,q}}$ and α, but not on τ_1,τ_2, and s_1, such that

$$\|g\|_{H^{\beta,p}}\le C.$$

Proof. Note that

$$\|g\|_{H^{\beta,p}} \le \int_{-1/2}^{1/2}\frac{\|g_w\|_{H^{\beta,p}}}{|w|^2}dw$$

where

$$g_w(u) := G^\alpha\left(\frac{\gamma(u+w)-\gamma(u)}{w}\right)\gamma'(u+s_1 w)|\gamma'(u+\tau_1 w)-\gamma(u+\tau_2 w)|^2.$$

Choosing some $\tilde p\in(p,p+1)$ which will be determined later on and letting $\tilde q := 2\frac{\tilde p p}{\tilde p - p}$ leads to

$$\frac{1}{p}=\frac{1}{2\tilde p}+\frac{1}{2\tilde p}+\frac{1}{\tilde q}+\frac{1}{\tilde q}.$$

Using that

$$\frac{\gamma(u+w)-\gamma(u)}{w}=\int_0^1\gamma'(u+\tau w)d\tau,$$

that γ is bi-Lipschitz, and that G^α is analytic away from the origin, we get according to the fractional chain rule (Lemma 4.2)

$$\left\|G^\alpha\left(\frac{\gamma(\cdot+w)-\gamma(\cdot)}{w}\right)\right\|_{H^{\beta,\tilde q}}\le C\|\gamma\|_{H^{\beta+1,\tilde q}}\le C.$$

Using the fractional Leibniz rule (Lemma 4.1), we derive

$$\begin{aligned}\|g_w\|_{H^{\beta,p}} &\le C\left\|G^\alpha\left(\frac{\gamma(\cdot+w)-\gamma(\cdot)}{w}\right)\right\|_{H^{\beta,\tilde q}}\|\gamma'\|_{H^{\beta,\tilde q}}\|\gamma'(\cdot+\tau_1 w)-\gamma'(\cdot+\tau_2 w)\|^2_{H^{\beta,2\tilde p}}\\ &\le C\left(\|\gamma'(\cdot+\tau_1 w)-\gamma'(\cdot+\tau_2 w)\|^2_{L^{2\tilde p}}+\||D|^{\beta+1}\gamma(\cdot+\tau_1 w)-|D|^{\beta+1}\gamma(\cdot+\tau_2 w)\|^2_{L^{2\tilde p}}\right).\end{aligned}$$

Hence,

$$\begin{aligned}\|g\|_{H^{\beta,p}} &\le C\int_{-1/2}^{1/2}\frac{\|\gamma'(\cdot+\tau_1 w)-\gamma'(\cdot+\tau_2 w)\|^2_{L^{2\tilde p}}}{w^2}dw\\ &\quad+C\int_{-1/2}^{1/2}\frac{\||D|^{\beta+1}\gamma(\cdot+\tau_1 w)-|D|^{\beta+1}\gamma(\cdot+\tau_2 w)\|^2_{L^{2\tilde p}}}{w^2}dw\\ &\le C\int_{-1/2}^{1/2}\frac{\|\gamma'(\cdot)-\gamma'(\cdot+(\tau_2-\tau_1)w)\|^2_{L^{2\tilde p}}}{w^2}dw\\ &\quad+C\int_{-1/2}^{1/2}\frac{\||D|^{\beta+1}\gamma(\cdot)-|D|^{\beta+1}\gamma(\cdot+(\tau_2-\tau_1)w)\|^2_{L^{2\tilde p}}}{w^2}dw\\ &\le C|\tau_2-\tau_1|\int_{-1}^{1}\frac{\|\gamma'(\cdot)-\gamma'(\cdot+w)\|^2_{L^{2\tilde p}}}{w^2}dw\\ &\quad+C|\tau_2-\tau_1|\int_{-1}^{1}\frac{\||D|^{\beta+1}\gamma(\cdot)-|D|^{\beta+1}\gamma(\cdot+w)\|^2_{L^{2\tilde p}}}{w^2}dw\\ &\le C\|\gamma'\|^2_{B^{1/2}_{2\tilde p,2}}+C\||D|^{\beta+1}\gamma\|^2_{B^{1/2}_{2\tilde p,2}}\le C\end{aligned}$$

if $\tilde p\in(p,p+1)$ is chosen so small that

$$\beta_0-\frac{1}{q}>\beta-\frac{1}{2\tilde p}>\beta-\frac{1}{2p}.$$

This proves Lemma 4.3. □

We use the last lemma to prove

Corollary 4.4. *Let γ,β_0,β,p and q be as in Lemma 4.3 and p' be such that $1/p+1/p'=1$. Then*

- *for all $\alpha \geq 2$ there is a constant C such that*

$$|T^{\alpha}_{s_1,s_2,\tau_1,\tau_2}(h)| \leq C\|h\|_{H^{1-\beta,p'}}$$

  for all $s_1, s_2, \tau_1, \tau_2 \in [0,1]$ and $h \in C^{\infty}$,
- *the operator $T(\gamma,\cdot) = T_1(\gamma,\cdot) + T_2(\gamma,\cdot) \in \left(H^{3/2}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)\right)^*$ can be extended to a bounded operator on $H^{1-\beta,p'}$.*

Proof. For $h \in C^{\infty}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ we have

$$T(\gamma,h) = -\int_0^1\int_0^1 T^2_{0,0,t_1,t_2}(h)dt_1dt_2 + \int_0^1\int_0^1\int_0^1\int_0^1 T^4_{s_1,s_2,t_1,t_2}(h)ds_1ds_2dt_1dt_2$$

and hence the second part is an immediate consequence of the first one.

Let $\Lambda^s := (\mathrm{id} - \Delta)^{\frac{s}{2}}$. Using that Λ^{β} is self adjoint, we get

$$\begin{aligned} T^{\alpha}_{s_1,s_2,\tau_1,\tau_2}(h) &= \int_{-1/2}^{1/2}\int_{\mathbb{R}/\mathbb{Z}} G^{\alpha}\left(\frac{\gamma(\cdot+w)-\gamma(\cdot)}{w}\right)\frac{|\gamma'(u+\tau_1 w)-\gamma'(u+\tau_2 w)|^2}{w^2} \\ &\qquad\qquad \gamma'(u+s_1w)h'(u+s_2w)dudw \\ &= \int_{\mathbb{R}/\mathbb{Z}}\int_{-1/2}^{1/2} \frac{\left(\Lambda^{\beta}g_w\right)(u)}{w^2}\left(\Lambda^{-\beta}h'\right)(u+s_2w)dwdu \end{aligned}$$

and hence as in the proof of Lemma 4.3

$$|T_{s_1,s_2,\tau_1,\tau_2}(h)| \leq C\int_{-1/2}^{1/2}\frac{\|g_w\|_{H^{\beta},p}}{w^2}\,\mathrm{d}w\,\|h\|_{H^{1-\beta,p'}} \leq C\,\|h\|_{H^{1-\beta,p'}}$$

where $C < \infty$ as in Lemma 4.3 does not depend on on s_1, s_2, τ_1,or τ_2. □

Using the two statements above, we are led to the following fact from which Theorem IV immediately follows.

Lemma 4.5. *Let $\gamma \in H^{3/2+\beta_0,q}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$, $\beta_0 \geq 0$, $q \in [2,\infty)$ and $\beta_0 - \frac{1}{q} > -1/2$ be a stationary point of the Möbius energy parametrized by arc length. Then,*

- *if $\beta_0 = 0$, we have $\gamma \in H^s$ for all $s < 3/2 + 2(1/2 - 1/q)$,*
- *if $0 < \beta_0 < 1/2$, we have $\gamma \in H^s$ for all $s < 3/2 + 2(\beta_0 + 1/2 - 1/q)$,*
- *if $\beta_0 \geq 1/2$, we have $\gamma \in H^{3/2+\beta_0+1/4}$.*

Proof. We set

$$\begin{aligned}
&\beta = 0, && \frac{1}{p} = \frac{2}{q} + \varepsilon, && \text{if } \beta_0 = 0,\\
&\beta = 0, && \frac{1}{p} = \frac{2}{q} - 2\beta_0 + \varepsilon, && \text{if } 0 < \beta_0 < 1/2,\\
&\beta = \beta_0 - 1/4, && \frac{1}{p} = 2/3, && \text{if } \beta_0 \geq \frac{1}{2}
\end{aligned}$$

and see that in each case the exponents satisfy the assumptions for the preceding Corollary 4.4 for all small enough $\varepsilon > 0$, so, for p' with $\frac{1}{p} + \frac{1}{p'} = 1$, the functional $T(\gamma, \cdot) = T_1(\gamma, \cdot) + T_2(\gamma, \cdot)$ can be extended to an operator in $\left(H^{1-\beta,p'}\right)^* \subset (H^{3/2-1/p'-\beta})^*$.

From the fact that γ is a stationary point of the Möbius energy we then deduce that

$$Q(\gamma, \cdot) \in (H^{3/2-1/p'-\beta})^*$$

and a comparison of the Fourier coefficients gives

$$\gamma \in H^{3/2+\beta+\frac{1}{p'}}.$$

Since

$$3/2 + \beta + \frac{1}{p'} = 5/2 + \beta - \frac{1}{p} = \begin{cases} 3/2 + 2(1/2 - 1/q) - \varepsilon & \text{if } \beta_0 = 0,\\ 3/2 + 2(\beta_0 + 1/2 - 1/q) - \varepsilon & \text{if } 0 < \beta_0 < 1/2,\\ 3/2 + \beta_0 + 1/4 & \text{else,} \end{cases}$$

this proves Lemma 4.5. □

A. Appendix

In this section we gather some facts most of which can already be found in [Sch11] in slightly different versions. The main aim is to prove Lemma 3.1 and 3.2 which both rely on quasi-locality of the Riesz potential I_s. Afterwards, we give an easy proof of the iteration lemma needed to deduce Dirichlet growth.

We will mainly deal with functions belonging to the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing smooth functions $\mathbb{R} \to \mathbb{R}$. The statements carry over to more general situations by suitable approximation arguments.

A.1. Quasi-locality

The essential tool apart from Sobolev inequalities is the following quantitative version of the *quasi-locality* of the fractional Laplacian and the Riesz potential.

Lemma A.1 (Quasi-locality). *Let $p_1, p_2, q_1, q_2 \in [1,\infty]$, where we assume that $q_2 = 1$ if $p_2 = 1$, $s \in (-1,1)$ and Ω_1, Ω_2 be disjoint domains with $d := \operatorname{dist}(\Omega_1,\Omega_2) > 0$ and with positive and finite Lebesgue measure. Then, for any $f \in \mathcal{S}(\mathbb{R})$,*

$$\|\Delta^{\frac{s}{2}}(f\chi_{\Omega_2})\|_{(p_1,q_1),\Omega_1} \lesssim d^{-1-s}|\Omega_1|^{1/p_1}|\Omega_2|^{1-1/p_2}\|f\|_{(p_2,q_2),\Omega_2}$$

where we set

$$\Delta^{\frac{s}{2}} := \begin{cases} |D|^s & \text{if } s > 0, \\ \mathrm{id} & \text{if } s = 0, \\ I_{|s|} & \text{if } s < 0. \end{cases}$$

Note, that $\|\cdot\|_{(1,q_1)}$, $\|\cdot\|_{(\infty,q_\infty)}$ are only considered in the inequalities that follow for $q_1 = 1$ and $q_\infty = \infty$.

Proof. For $\tilde{k}_s(z) = \frac{1}{|z|^{1+s}}\chi_{\mathbb{R}\setminus B_d}(z)$ and $\operatorname{supp} f \subset \Omega_2$ we have for all $x \in \Omega_1$

$$\Delta^{\frac{s}{2}} f(x) = c_s(\tilde{k}_s * f)(x)$$

and hence

$$|\Delta^{\frac{s}{2}} f(x)| \lesssim \|f\|_{1,\Omega_2}\|\tilde{k}_s\|_\infty \le d^{-1-s}\|f\|_{1,\Omega_2}.$$

Hence,

$$\begin{aligned}\|\Delta^{\frac{s}{2}} f\|_{(p_1,q_1),\Omega_1} &\le |\Omega_1|^{1/p_1}\|\Delta^{\frac{s}{2}} f\|_{\infty,\Omega_1} \le d^{-1-s}|\Omega_1|^{1/p_1}\|f\|_{1,\Omega_2} \\ &\le d^{-1-s}|\Omega_1|^{1/p_1}|\Omega_2|^{1-1/p_2}\|f\|_{(p_2,q_2),\Omega_2}.\end{aligned}$$

□

A quite immediate consequence of this quasi-locality and Sobolev imbeddings is the following lemma. To state it, let

$$A^k_{\Lambda,r} := B_{2^k\Lambda r} - B_{2^{k-1}\Lambda r}. \tag{A.1}$$

Lemma A.2. *Let $p \in (1,\infty), q, \tilde{q} \in [1,\infty]$, $s \in (-1, 1/p)$ and $-\frac{1}{p^*} = s - \frac{1}{p}$.*

(i) *For $f \in \mathcal{S}(\mathbb{R})$ with $\operatorname{supp} f \subset B_r$ we have*

$$\|\Delta^{-\frac{s}{2}} f\|_{(p^*,q);\mathbb{R}-B_{\Lambda r}} \lesssim \Lambda^{-1+\frac{1}{p}}\|f\|_{(p,\tilde{q}),B_r}$$

uniformly for all $\Lambda > 2$.

(ii) *If $s \in [0, \frac{1}{p})$, we have for all $f \in \mathcal{S}(\mathbb{R})$*

$$\|I_s f\|_{(p^*,q);\mathbb{R}-B_{\Lambda r}} \lesssim \Lambda^{-1+\frac{1}{p}}\|f\|_{(p,\tilde{q});B_r} + \|f\|_{(p,q);\mathbb{R}-B_r}$$

uniformly for all $\Lambda > 2$.

(iii) *For $s \in [0, \frac{1}{p})$ and any $f \in \mathcal{S}(\mathbb{R})$, we have*

$$\|I_s f\|_{(p^*,q),B_r} \lesssim \|f\|_{(p,q),B_{\Lambda r}} + \Lambda^{-\frac{1}{p^*}} \sum_{k=1}^{\infty} \left(2^{-k}\right)^{\frac{1}{p^*}} \|f\|_{(p,\tilde{q}),A^k_{\Lambda,r}}$$

uniformly for all $\Lambda > 2$.

Proof. For the first inequality we use Lemma A.1 and sum up the estimate

$$\|\Delta^{-\frac{s}{2}} f\|_{(p^*,q),A^k_{\Lambda,r}} \lesssim (\Lambda 2^k)^{-1+\frac{1}{p}} \|f\|_{(p,\tilde{q});B_r}.$$

In the second case, we use

$$\|I_s f\|_{(p^*,q),\mathbb{R}-B_{\Lambda r}} \leq \|I_s(\chi_{B_r} f)\|_{(p^*,q),\mathbb{R}-B_{\Lambda r}} + \|I_s((1-\chi_{B_r})f)\|_{(p^*,q),\mathbb{R}-B_{\Lambda r}}$$

and estimate the first term using (i) and the second term using Sobolev's inequality to get

$$\|I_s f\|_{(p^*,q),\mathbb{R}-B_{\Lambda r}} \lesssim \Lambda^{-1+\frac{1}{p}} \|f\|_{(p,\tilde{q}),B_r} + \|f\|_{(p,q);\mathbb{R}-B_r}.$$

To deduce the last inequality, we decompose $f = \chi_{B_{\Lambda r}} f + \sum_{k=1}^{\infty} \chi_{A^k_{\Lambda,r}} f$ and estimate using Sobolev's inequality

$$\|I_s(\chi_{B_{\Lambda r}} f)\|_{(p^*,q),B_r} \lesssim \|f\|_{(p,q),B_{\Lambda r}}$$

and using Lemma A.1

$$\|I_s(\chi_{A^k_{\Lambda,r}} f)\|_{(p^*,q),B_r} \lesssim (\Lambda 2^k)^{-\frac{1}{p^*}} \|f\|_{(p,\tilde{q}),A^k_{\Lambda,r}}.$$

Summing up, this proves the last inequality. □

Finally, we use the quasi-locality to prove

Proposition A.3. *For $p \in (1,\infty)$, $q \in [1,\infty]$, $s,t \geq 0$ with $0 < s+t < 1$ there is a $\theta > 0$ such that we have for any $f \in L^{p,q}(\mathbb{R})$, $\Lambda > 2$, and $r > 0$*

$$\||D|^s f\|_{(p,q),B_r} \lesssim \sup_{\substack{\varphi \in C_0^\infty(B_{\Lambda 2_r}),\\ \||D|^t \varphi\|_{(p',q')} \leq 1}} \int f|D|^{s+t}\varphi + r^{-s}\Lambda^{-\theta} \|f\|_{(p,q);B_{\Lambda r}} + r^{-s}\Lambda^{-\theta} \sum_{l=1}^{\infty} 2^{-\theta l} \|f\|_{(p,q),A^l_{\Lambda,r}}.$$

Proof. Assume that

$$\sup_{\varphi \in C_0^\infty(B_{\Lambda 2_r}), \||D|^t \varphi\|_{(p',q')} \leq 1} \int f|D|^{s+t}\varphi \leq K.$$

For $g \in C_0^\infty(B_r)$ we consider

$$\int_{\mathbb{R}} (|D|^s f) g dx = \int_{\mathbb{R}} f |D|^s g dx = \int_{B_{\Lambda r}} f |D|^s g dx + \sum_{l=1}^{\infty} \int_{A^l_{\Lambda,r}} f |D|^s g dx.$$

For the first term we use a smooth partition of unity $(\eta_k)_{k\in\mathbb{N}\cup\{0\}}$ with $\operatorname{supp}\eta_0 \subset B_{\Lambda^2 r}$ and $\operatorname{supp}\eta_k \subset B_{(\Lambda^2 2^k+1)r} - B_{(\Lambda^2 2^{k-1}-1)r}$ to get

$$\begin{aligned}\left|\int_{B_{\Lambda r}} f|D|^s g dx\right| &\le \left|\int_{B_{\Lambda r}} f|D|^{t+s}(\eta_0 I_t g)\,dx\right| + \sum_{l=1}^{\infty}\left|\int_{B_{\Lambda r}} f|D|^{t+s}(\eta_l I_t g)\,dx\right| \\ &\lesssim K\|g\|_{(p',q')} + \|f\|_{(p,q);B_{\Lambda r}} \sum_{l=1}^{\infty} \||D|^{t+s}(\eta_l I_t g)\|_{(p',q');B_{\Lambda r}}.\end{aligned}$$

Since by Lemma A.1 we have

$$\begin{aligned}\||D|^{t+s}\eta_l I_t g\|_{(p',q');B_{\Lambda r}} &\lesssim (\Lambda^2 2^l r)^{-1-t-s}(\Lambda r)^{1/p'}(\Lambda^2 2^l r)^{1-1/p'}\|\eta_l I_t g\|_{(p',q')} \\ &\le (2^l)^{-t-s-1/p'}\Lambda^{-(2s+2t+1/p')} r^{-s-t}\|I_t g\|_{(p',q')}\end{aligned}$$

we can estimate this further by

$$K\|g\|_{(p',q')} + \Lambda^{-\theta_1} r^{-s-t}\|f\|_{(p,q);B_{\Lambda r}}\|I_t g\|_{(p',q')}$$

for $\theta_1 = 2t + 2s + 1/p'$.

Since $\operatorname{supp} g \subset B_r$ we get a scaled Poincaré inequality by applying first the Sobolev and then the Hölder inequality

$$\|I_t g\|_{(p',q')} \lesssim \|g\|_{(\frac{p'}{p't+1},q'),B_r} \lesssim r^t \|g\|_{(p',q')}$$

and using the quasi-locality (Lemma A.1) once more

$$\||D|^s g\|_{(p',q'),A^k_{\Lambda r}} \lesssim (2^k\Lambda r)^{-1-s} r^{1-1/p'}(2^k\Lambda r)^{1/p'}\|g\|_{(p',q')} = \Lambda^{-\theta_2} 2^{-k\theta_2} r^{-s}\|g\|_{(p',q')}$$

for $\theta_2 = 1 + s - 1/p'$.

Hence,

$$\int_{\mathbb{R}} (|D|^s f)\ g dx \le \left(K + \Lambda^{-\theta} r^{-s}\|f\|_{(p,q);B_{\Lambda r}} + r^{-s}\Lambda^{-\theta}\sum_{l=1}^{\infty} 2^{-\theta l}\|f\|_{(p,q);A^l_{\Lambda,r}}\right)\|g\|_{(p',q')},$$

for $\theta = \min\{\theta_1, \theta_2\}$ which by duality proves the proposition. □

A.2. Proofs of Lemmata 3.1 and 3.2

The following lemma is the starting point for the estimates of H_s and essentially follows from the mean value theorem or a first-order Taylor expansion.

Lemma A.4. *Let $\delta \in [0,1]$, $\alpha \in (0,1)$. Then for almost all $x, y, \xi \in \mathbb{R}^n$ we have*

$$\left||x-\xi|^{-1+\alpha} - |y-\xi|^{-1+\alpha}\right| \lesssim |x-y|^{\delta}\left(|y-\xi|^{-1+\alpha-\delta} + |x-\xi|^{-1+\alpha-\delta}\chi_{|x-y|>2|x-\xi|}\right).$$

Proof. If $|x-y| > 2|x-\xi|$ we get

$$|y-\xi| \geq |y-x| - |x-\xi| > |x-\xi|$$

and hence

$$\left||x-\xi|^{-1+\alpha} - |y-\xi|^{-1+\alpha}\right| \lesssim |x-\xi|^{-1+\alpha} \lesssim |x-y|^{\delta}|x-\xi|^{-1+\alpha-\delta}.$$

If $|x-y| \leq 2|x-\xi|$ we first observe that the above argument leads to

$$\left||x-\xi|^{-1+\alpha} - |y-\xi|^{-1+\alpha}\right| \lesssim |x-y|^{\delta}|y-\xi|^{-1+\alpha-\delta}$$

if $|x-y| > 2|y-\xi|$.

To deal with the case that both $|x-y| \leq 2|x-\xi|$ and $|x-y| \leq 2|y-\xi|$ we observe that then

$$|y-\xi| \leq |x-\xi| + |x-y| \leq 3|x-\xi|.$$

Hence, we get using the mean value theorem

$$\begin{aligned}\left||x-\xi|^{-1+\alpha} - |y-\xi|^{-1+\alpha}\right| &\lesssim |x-y| \max\left\{|x-\xi|^{-2+\alpha}, |y-\xi|^{-2+\alpha}\right\} \\ &\lesssim |x-y||y-\xi|^{-2+\alpha} \lesssim |x-y|^{\delta}|y-\xi|^{-1+\alpha-\delta}.\end{aligned}$$

□

We use the lemma above to derive the following pointwise estimate for $H_{1/2+s}$.

Lemma A.5 ([Sch11]). *For $s \in [0, \frac{1}{2})$ and functions $a, b \in \mathcal{S}(\mathbb{R})$ the following holds for any $\varepsilon, \varepsilon' \in [0, \frac{1}{6} - \frac{s}{3})$:*

$$\begin{aligned}|H_{1/2+s}(a,b)| \lesssim\ & I_{1/6-s/3}(I_{1/6-s/3-\varepsilon}||D|^{1/2-\varepsilon}a|\ I_{1/6-s/3-\varepsilon'}||D|^{1/2-\varepsilon'}b|) \\ &+ I_{1/3-2s/3-\varepsilon}||D|^{1/2-\varepsilon}a|\ I_{1/6-s/3-\varepsilon'}||D|^{1/2-\varepsilon'}b| \\ &+ I_{1/6-s/3-\varepsilon}||D|^{1/2-\varepsilon}a|\ I_{1/3-2s/3-\varepsilon'}||D|^{1/2-\varepsilon'}b| \\ &+ I_{1/4-s/2-\varepsilon}||D|^{1/2-\varepsilon}a|\ I_{1/4-s/2-\varepsilon'}||D|^{1/2-\varepsilon'}b|.\end{aligned}$$

Proof. In order to shorten notation, we restrict to $\varepsilon' = \varepsilon$; the general case is parallel. We use the identities $a = I_{\frac{1}{2}-\varepsilon}|D|^{\frac{1}{2}-\varepsilon}a$ and $b = I_{\frac{1}{2}-\varepsilon}|D|^{\frac{1}{2}-\varepsilon}b$. Then,

$$\begin{aligned}
&|H_{1/2+s}(a,b)(x)| \\
&\lesssim \int\int\int \frac{\left(|y-z|^{-1+1/2-\varepsilon} - |x-z|^{-1+1/2-\varepsilon}\right)\left(|y-w|^{-1+1/2-\varepsilon} - |x-w|^{-1+1/2-\varepsilon}\right)}{|y-x|^{1+(1/2+s)}} \cdot \\
&\qquad \cdot \left||D|^{\frac{1-2\varepsilon}{2}}a(z)\right| \left||D|^{\frac{1-2\varepsilon}{2}}b(w)\right| \mathrm{d}w\,\mathrm{d}z\,\mathrm{d}y.
\end{aligned}$$

Applying Lemma A.4 we get

$$\begin{aligned}
&\frac{\left(|y-z|^{-1+1/2-\varepsilon} - |x-z|^{-1+1/2-\varepsilon}\right)\left(|y-w|^{-1+1/2-\varepsilon} - |x-w|^{-1+1/2-\varepsilon}\right)}{|y-x|^{1+(1/2+s)}} \\
&\lesssim \frac{\left(|y-w|^{-1+1/2-\varepsilon-\delta}+|x-w|^{-1+1/2-\varepsilon-\delta}\chi_{|x-y|>2|x-w|}\right)\left(|y-z|^{-1+1/2-\varepsilon-\delta}+|x-z|^{-1+1/2-\varepsilon-\delta}\chi_{|x-y|>2|x-z|}\right)}{|x-y|^{1+1/2+s-2\delta}} \\
&\lesssim \frac{(|y-w||y-z|)^{-1+1/2-\varepsilon-\delta} + (|x-w||y-z|)^{-1+1/2-\varepsilon-\delta} + (|y-w||x-z|)^{-1+1/2-\varepsilon-\delta}}{|x-y|^{1+1/2+s-2\delta}} \\
&\quad + \frac{(|x-w||x-z|)^{-1+1/2-\varepsilon-\delta}}{|x-y|^{1+1/2+s-2\delta}}\chi_{|x-y|>2|x-w|}\,\chi_{|x-y|>2|x-z|}.
\end{aligned}$$

For $\delta = 1/3 + s/3$ we hence get

$$\begin{aligned}
|H_{1/2+s}(a,b)| \lesssim\ & I_{1/6-s/3}(I_{1/6-s/3-\varepsilon}||D|^{\frac{1-2\varepsilon}{2}}a|\ I_{1/6-s/3-\varepsilon}||D|^{\frac{1-2\varepsilon}{2}}b|) \\
&+ I_{1/6-s/3-\varepsilon}||D|^{\frac{1-2\varepsilon}{2}}a|\ I_{1/3-2s/3-\varepsilon}||D|^{\frac{1-2\varepsilon}{2}}b| \\
&+ I_{1/3-2s/3-\varepsilon}||D|^{\frac{1-2\varepsilon}{2}}a|\ I_{1/6-s/3-\varepsilon}||D|^{\frac{1-2\varepsilon}{2}}b| + A
\end{aligned}$$

where

$$\begin{aligned}
A &\lesssim \int\int\left(\int \frac{(|x-w||x-z|)^{-1+1/2-\varepsilon-\delta}}{|x-y|^{1+1/2+s-2\delta}}\chi_{|x-y|>2|x-w|}\,\chi_{|x-y|>2|x-z|}\ \mathrm{d}y\right)||D|^{\frac{1-2\varepsilon}{2}}a(z)|\ ||D|^{\frac{1-2\varepsilon}{2}}b(w)|\,\mathrm{d}w\,\mathrm{d}z \\
&\lesssim \int\int |x-w|^{-1+1/4-\varepsilon-s/2}|x-z|^{-1+1/4-\varepsilon-s/2}||D|^{\frac{1-2\varepsilon}{2}}a(z)|\ ||D|^{\frac{1-2\varepsilon}{2}}b(w)|\,\mathrm{d}w\,\mathrm{d}z \\
&\lesssim I_{1/4-s/2-\varepsilon}||D|^{\frac{1-2\varepsilon}{2}}a|\ I_{1/4-s/2-\varepsilon}||D|^{\frac{1-2\varepsilon}{2}}b|.
\end{aligned}$$

□

To estimate this further, we will use the following fact about *lower order products* which we get using the quasi-locality.

Lemma A.6 (Lower order products). *Let for $s > 0$, $0 \le s_1, s_2, s_3 \le 1/2$, $s_1 + s_2 + s_3 = s$, and at least two of these s_i, $i = 1, 2, 3$, non-zero. Let $p \in (1, \infty)$, $p_2 \in (1, \frac{1}{s_2})$, $p_3 \in (1, \frac{1}{s_3})$ and such that*

$$\frac{1}{p} = \frac{1}{p_2} + \frac{1}{p_3} - s.$$

Assume moreover that

$$|G(u,v)| \lesssim I_{s_1}\left(I_{s_2}|u|\, I_{s_3}|v|\right),$$

then there is some $\theta > 0$ *such that we have for any* $\Lambda > 4$, *and* $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$,

$$\begin{aligned} &\|G(u,v)\|_{(p,q),B_r} \\ &\lesssim \|u\|_{(p_2,q_1),B_{\Lambda r}}\|v\|_{(p_3,q_2),B_{\Lambda r}} + \|u\|_{(p_2,q_1)}\Lambda^{-\theta}\sum_{k=2}^{\infty} 2^{-\theta k}\|v\|_{(p_3,q_2),B_{\Lambda 2^k r}}. \end{aligned} \tag{A.2}$$

If $\operatorname{supp}|D|^t v \subset \overline{B_r}$ *for some* $t \in [0, \frac{1}{2}]$, *we furthermore get for any* $k \geq 2$, $\Lambda > 16$,

$$\|G(u,v)\|_{(p,q),A^k_{\Lambda,r}} \lesssim \Lambda^{-\theta}2^{-\theta k}\|u\|_{(p_2,\infty),\mathbb{R}}\, \|v\|_{(p_3,1),\mathbb{R}}. \tag{A.3}$$

Proof. Let

$$\begin{aligned} \frac{1}{p_2^*} &:= \frac{1}{p_2} - s_2, \\ \frac{1}{p_3^*} &:= \frac{1}{p_3} - s_3, \\ \frac{1}{p_1} := \frac{1}{p_2^*} + \frac{1}{p_3^*} &= \frac{1}{p_2} + \frac{1}{p_3} - s_2 - s_3, \end{aligned}$$

and

$$\frac{1}{p_1^*} := \frac{1}{p} \equiv \frac{1}{p_2} + \frac{1}{p_3} - s = \frac{1}{p_1} - s_1.$$

Then by Lemma A.2 (iii), for $\Lambda \hat{=} \sqrt{\Lambda}$, $p^* \hat{=} p_1^* = p$, $p \hat{=} p_1$, $s \hat{=} s_1$

$$\begin{aligned} \|G(u,v)\|_{(p_1^*,q),B_r} &\lesssim \|\|I_{s_2}|u|\, I_{s_3}|v|\|\|_{(p_1,q),B_{\Lambda^{1/2}r}} + \Lambda^{-1/2p}\sum_{k=1}^{\infty} 2^{-k/p}\|\|I_{s_2}|u|\, I_{s_3}|v|\|\|_{(p_1,q),B_{\Lambda^{1/2}2^k r}} \\ &\lesssim \|\|I_{s_2}|u|\|\|_{(p_2^*,q_1),B_{\Lambda^{1/2}r}}\, \|\|I_{s_3}|v|\|\|_{(p_3^*,q_2),B_{\Lambda^{1/2}r}} + \Lambda^{-1/2p}\sum_{k=1}^{\infty} 2^{-k/p}\|\|I_{s_2}|u|\|\|_{(p_2^*,q_1),B_{2^k\Lambda^{1/2}r}}\, \|\|I_{s_3}|v|\|\|_{(p_3^*,q_2),B_{2^k\Lambda^{1/2}r}}. \end{aligned}$$

Applying yet again Lemma A.2 (iii) to both factors in each sum with $\Lambda \hat{=} \sqrt{\Lambda}$ we arrive at

$$\|G(u,v)\|_{(p,q),B_r} \lesssim \|u\|_{(p_2,q_1),B_{\Lambda r}}\|v\|_{(p_3,q_2),B_{\Lambda r}} + \|u\|_{(p_2,q_1)}\Lambda^{-\theta}\sum_{k=2}^{\infty} 2^{-\theta k}\|v\|_{(p_3,q_2),B_{\Lambda 2^k r}}.$$

As for the second estimate, we apply Lemma A.2 (ii) with $\Lambda \hat{=} \sqrt{\Lambda}$ to get

$$\|G(u,v)\|_{(p,q),\mathbb{R}-B_{\Lambda r}} = \|I_{s_1}\left(I_{s_2}|u|\, I_{s_3}|v|\right)\|_{(p_1^*,q),\mathbb{R}-B_{\Lambda r}}$$

$$\lesssim \Lambda^{-\theta_1} \||I_{s_2}|u|\ I_{s_3}|v|\|_{(p_1,\infty),B_{\Lambda^{1/2}r}} + \||I_{s_2}|u|\ I_{s_3}|v|\|_{(p_1,q),\mathbb{R}-B_{\Lambda^{1/2}r}}$$

where $\theta_1 = \frac{1}{2} - \frac{1}{2p_1}$. We estimate the first term using Hölder's inequality and Sobolev imbedding theorem to get

$$\Lambda^{-\theta_1} \||I_{s_2}|u|\ I_{s_3}|v|\|_{(p_1,\infty),B_{\Lambda^{1/2}r}} \leq \Lambda^{-\theta_1} \|u\|_{(p_2,\infty)} \|v\|_{(p_3,1)}.$$

For the second term we use Hölder's inequality and Sobolev's imbedding theorem to get

$$\begin{aligned} \||I_{s_2}|u|\ I_{s_3}|v|\|_{(p_1,q),\mathbb{R}-B_{\Lambda^{1/2}r}} &\lesssim \||I_{s_2}|u|\|_{(p_2^*,q_1),\mathbb{R}-B_{\Lambda^{1/2}r}} \||I_{s_3}|v|\|_{(p_3^*,q_2),\mathbb{R}-B_{\Lambda^{1/2}r}} \\ &\lesssim \|u\|_{(p_2,\infty),\mathbb{R}} \||I_{s_3}|v|\|_{(p_3^*,q_2),\mathbb{R}-B_{\Lambda^{1/2}r}}. \end{aligned}$$

Since

$$\||I_{s_3}|v|\|_{(p_3^*,q_2),\mathbb{R}-B_{\Lambda^{1/2}r}} \lesssim \Lambda^{-\theta_2} \|v\|_{(p_3,1);B_{\Lambda^{1/4}}} + \|v\|_{(p_3,q);\mathbb{R}-B_{\Lambda^{1/4}}}$$

where $\theta_2 = \frac{1}{4} - \frac{1}{4p_3}$ and, using Lemma A.2 (i)

$$\|v\|_{(p_3,q);\mathbb{R}-B_{\Lambda^{1/4}}} = \||D|^t (I_t v)\|_{(p_3,q);\mathbb{R}-B_{\Lambda^{1/4}}} \leq \Lambda^{-\theta_2} \|I_t v\|_{(p_3^{**},1)} \leq \Lambda^{-\theta_2} \|I_t v\|_{(p_3,1)}$$

we deduce the statement for $\theta := \min\{\theta_1, \theta_2\}$. □

We then have the following

Lemma A.7. *There is a $\varepsilon_0 > 0$ such that for $\delta, \varepsilon \in [0, \varepsilon_0)$, for any $a, b \in \mathcal{S}(\mathbb{R})$, $\Lambda > 4$*

$$\begin{aligned} \|H_{\frac{1}{2}}(a,b)\|_{(2,q),B_r} \lesssim\ & \||D|^{1/2-\delta} a\|_{(\frac{2}{1-2\delta},q_1),B_{\Lambda r}} \||D|^{1/2-\varepsilon} b\|_{(\frac{2}{1-2\varepsilon},q_2),B_{\Lambda r}} + \\ &+ \||D|^{1/2-\delta} a\|_{(\frac{2}{1-2\delta},q_1)} \Lambda^{-\theta} \sum_{k=1}^{\infty} 2^{-\theta(k-1)} \||D|^{1/2-\varepsilon} b\|_{(\frac{2}{1-2\varepsilon},q_2),B_{2^k\Lambda r}}. \end{aligned}$$

If $\operatorname{supp} b \subset B_r$ *we furthermore get for any $k \geq 2$, $\Lambda > 16$, $s \in [0, \frac{1}{2})$*

$$\|H_{\frac{1}{2}}(a,b)\|_{(\frac{2}{1+2s},q),A^k_{\Lambda,r}} \lesssim (\Lambda 2^k)^{-\theta} \||D|^{\frac{1}{2}} a\|_{(2,\infty),\mathbb{R}}\ \||D|^{1/2-\varepsilon} b\|_{(\frac{2}{1-2\varepsilon},1),\mathbb{R}}.$$

Proof. Immediately from Lemma A.5, Lemma A.6 where $s = \frac{1}{2} - \varepsilon - \delta$. □

We use the lemma above to estimate the normal part as stated in Lemma 3.1.

Proof of Lemma 3.1. With (3.10) we obtain $\left\||D|^s |g'|^2\right\|_{(\frac{2}{1+2s},\infty),B_r} \lesssim r^{\frac{1}{2}+s}$. In order to show (3.11) by the decomposition (3.8) it remains to treat the H-term. We rewrite

$$|D|^s H_{\frac{1}{2}}(a,b) \quad = \quad \underbrace{H_{s+\frac{1}{2}}(a,b)}_{=:I} + \underbrace{a|D|^{\frac{1+2s}{2}} b - |D|^s (a|D|^{\frac{1}{2}} b)}_{=:II} + \underbrace{b|D|^{\frac{1+2s}{2}} a - |D|^s (b|D|^{\frac{1}{2}} a)}_{=:III}.$$

We will show that all three terms satisfy the hypotheses of Lemma A.6. Due to Lemma A.5 this is true for the term I.

As for II note that $II = 0$ in case $s = 0$. If $s \in (0,1)$, the potential definition (3.2) of $|D|^s$ gives

$$\begin{aligned} II(x) &= c\left(a(x)\int \frac{|D|^{\frac12}b(y) - |D|^{\frac12}b(x)}{|x-y|^{1+s}}\,\mathrm{d}y - \int \frac{a(y)|D|^{\frac12}b(y) - a(x)|D|^{\frac12}b(x)}{|x-y|^{1+s}}\,\mathrm{d}y\right)\\ &= c\int \frac{(a(x)-a(y))\,|D|^{\frac12}b(y)}{|x-y|^{1+s}}\,\mathrm{d}y. \end{aligned}$$

Using $a = I_{\frac12}|D|^{\frac12}a$ we arrive at

$$|II(x)| \lesssim \int \frac{|a(y)-a(x)|\left||D|^{\frac12}b(y)\right|}{|x-y|^{1+s}}\,\mathrm{d}y \lesssim \int\int \frac{\left||y-z|^{-1+\frac12} - |x-z|^{-1+\frac12}\right|\left||D|^{\frac12}b(y)\right|\left||D|^{\frac12}a(z)\right|}{|x-y|^{1+s}}\,\mathrm{d}y\,\mathrm{d}z.$$

For almost all x, y, z we get from Lemma A.4 choosing $\delta := \frac14 + \frac{s}{2}$

$$\frac{\left||y-z|^{-1+\frac12} - |x-z|^{-1+\frac12}\right|}{|x-y|^{1+s}} \le (|x-z||x-y|)^{-1+\frac14-\frac{s}{2}} + (|y-z||x-y|)^{-1+\frac14-\frac{s}{2}},$$

which implies, using again (3.2)

$$|II(x)| \lesssim I_{\frac{1-2s}{4}}\left||D|^{\frac12}a\right| I_{\frac{1-2s}{4}}\left||D|^{\frac12}b\right| + I_{\frac{1-2s}{4}}\left(\left||D|^{\frac12}b\right| I_{\frac{1-2s}{4}}\left||D|^{\frac12}a\right|\right).$$

By symmetry a respective estimate holds also for the term III. Applying Lemma A.6 one concludes. □

Proof of Lemma 3.2. Again the proof relies on quasi-locality. First we decompose

$$\begin{aligned} \left|\int |D|^{\frac12}g_i'\omega_{ij}H_{1/2}(g_j',\phi)\right| &\lesssim \left|\int\limits_{B_{\Lambda^{1/2}r}} |D|^{\frac12}g_i'\omega_{ij}H_{1/2}(g_j',\phi)\right| + \sum_{k=1}^{\infty}\left|\int\limits_{A^k_{\Lambda^{1/2},r}} |D|^{\frac12}g_i'\omega_{ij}H_{1/2}(g_j',\phi)\right|\\ &\lesssim \||D|^{\frac12}g'\|_{(2,\infty),B_{\Lambda^{1/2}r}}\|H_{1/2}(g_j',\phi)\|_{(2,1),B_{\Lambda^{1/2}r}}\\ &\quad + \sum_{k=1}^{\infty}\||D|^{\frac12}g'\|_{(2,\infty),A^k_{\Lambda^{1/2},r}}\|H_{1/2}(g_j',\phi)\|_{(2,1),A^k_{\Lambda^{1/2},r}}. \end{aligned}$$

Using Lemma A.7, the first summand can be estimated by

$$\|I_s|D|^{\frac12}\phi\|_{(\frac{2}{1-2s},1)}\left(\||D|^{\frac12}g'\|^2_{(2,\infty),B_{\Lambda r}} + \||D|^{\frac12}g'\|_{(2,\infty),B_{\Lambda r}}\,\Lambda^{-\theta}\sum_{l=1}^{\infty}2^{-\theta l}\||D|^{\frac12}g'\|_{(2,\infty),B_{\Lambda^2 l_r}}\right).$$

Applying the second part of Lemma A.7, the infinite sum can be estimated by

$$\Lambda^{-\theta}\||I_s|D|^{\frac{1}{2}}\phi\|_{(\frac{2}{1-2s},1)}\,\||D|^{\frac{1}{2}}g'\|_{(2,\infty)}\sum_{k=2}^{\infty}2^{-\theta k}\||D|^{\frac{1}{2}}g'\|_{(2,\infty),B_{2^k\Lambda r}}.$$

□

A.3. Iteration lemma

In order to prove Dirichlet growth, we need an iteration lemma whose proof is based on the technique presented in [Tao01, notes4, p. 11]. The statement is also similar to the corresponding one appearing in [DLR11a]. One should see this as a generalized version of De Giorgi's Iteration Lemma, cf., e.g., [Gia83].

Lemma A.8 (Iteration lemma). *Let $C<\infty$ and $\theta>0$ be given.*

If $b_k\geq 0$, $k\in\mathbb{N}_0$, satisfy

$$b_{k+m}\leq \varepsilon b_k + C\left(2^{-\theta(k+m)}+2^{-\theta m}\sum_{l=1}^{k}2^{-\theta l}b_{k-l}\right) \tag{A.4}$$

for all $k\in\mathbb{N}_0$ and $\varepsilon>0$ is small enough, m is big enough, then

$$b_k\lesssim 2^{-\tilde{\theta}k}$$

for all $k\in\mathbb{N}_0$ with $\tilde{\theta}=\theta/2$.

Proof. We will prove that

$$\sum_{k=0}^{\infty}2^{\tilde{\theta}k}b_k\leq C\sum_{l=0}^{m}2^{\tilde{\theta}l}b_l+C. \tag{A.5}$$

Especially, the infinite sum converges and hence the summands are a null series, which proves the lemma.

Multiplying Equation (A.4) with $2^{\tilde{\theta}k}$, and summing over k we get

$$\begin{aligned}
2^{-\tilde{\theta}m}\sum_{k=0}^{\infty}2^{\tilde{\theta}(k+m)}b_{k+m} &\leq \varepsilon\sum_{k=0}^{\infty}2^{\tilde{\theta}k}b_k + C\sum_{k=0}^{\infty}2^{-\theta(k+m)}2^{\tilde{\theta}k}+C2^{-\theta m}\sum_{k=0}^{\infty}\sum_{l=1}^{k}2^{-\theta l}2^{\tilde{\theta}k}b_{k-l}\\
&\leq \varepsilon\sum_{k=0}^{\infty}2^{\tilde{\theta}k}b_k + C2^{-\theta m}\sum_{k'=0}^{\infty}\sum_{l=1}^{\infty}2^{-\theta l}2^{\tilde{\theta}(k'+l)}b_{k'}+C2^{-\theta m}\\
&\leq \varepsilon\sum_{k=0}^{\infty}2^{\tilde{\theta}k}b_k + \tilde{C}2^{-\theta m}\sum_{k'=0}^{\infty}2^{\tilde{\theta}k'}b_{k'}+C2^{-\theta m}.
\end{aligned}$$

Assuming that m is so large that $\tilde{C}2^{-m(\theta-\tilde{\theta})} = \tilde{C}2^{-\frac{m\theta}{2}} < 1/4$ and $0 < \varepsilon < \frac{1}{4}2^{-\theta m}$ we get

$$\sum_{k=0}^{\infty} 2^{\tilde{\theta}k} b_k \leq \frac{1}{2} \sum_{k=0}^{\infty} 2^{\tilde{\theta}k} b_k + \sum_{l=0}^{m} 2^{\tilde{\theta}l} b_l + C$$

and hence

$$\sum_{k=0}^{\infty} 2^{\tilde{\theta}k} b_k \leq C \sum_{l=0}^{m} 2^{\tilde{\theta}l} b_l + C$$

if the infinite series converges. If the sum is not known to converge, we apply the above argument to the cut-off series

$$\begin{cases} \tilde{b}_k = b_k & \text{if } k \leq N, \\ \tilde{b}_k = 0 & \text{else} \end{cases}$$

and get the uniform bound $\sum_{k=0}^{N} 2^{\tilde{\theta}k} b_k \lesssim \sum_{l=0}^{m} 2^{\tilde{\theta}l} b_l + 1$. Letting $N \to \infty$ we get (A.5). □

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