Regularity of n/2-harmonic maps into spheres

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We prove Hölder continuity for $n/2$-harmonic maps from subsets of $\mathbb{R}^n$ into a sphere. This extends a recent one-dimensional result by F. Da Lio and T. Rivière to arbitrary dimensions. The proof relies on compensation effects which we quantify adapting an approach for Wente's inequality by L. Tartar, instead of Besov-space arguments which were used in the one-dimensional case. Moreover, fractional analogues of Hodge decomposition and higher order Poincaré inequalities as well as several localization effects for nonlocal operators similar to the fractional laplacian are developed and applied.

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Contents

1 Introduction

In his seminal work [Hél90] F. Hélein proved regularity for harmonic maps from the two-dimensional unit disk $B_1(0) \subset \mathbb{R}^2$ into the m-dimensional sphere $\mathbb{S}^{m-1} \subset \mathbb{R}^m$ for arbitrary $m \in \mathbb{N}$. These maps are critical points of the functional

$$E_2(u) := \int\limits_{B_1(0)\subset\mathbb{R}^2} |\nabla u|^2, \qquad \text{where } u \in W^{1,2}(B_1(0), \mathbb{S}^{m-1}).$$

The importance of this result is the fact that harmonic maps in two dimensions are special cases of critical points of conformally invariant variational functionals, which play an important role in physics and geometry and have been studied for a long time: Hélein's approach is based on the discovery of a compensation phenomenon appearing in the Euler-Lagrange equations of E_2, using a relation between div-curl expressions and the Hardy space. This kind of relation had been discovered shortly before in the special case of determinants by S. Müller [Mül90] and was generalized by R. Coifman, P.L. Lions, Y. Meyer and S. Semmes [CLMS93]. Hélein extended his result to the case where the sphere $\mathbb{S}^{m-1}$ is replaced by a general target manifold developing the so-called moving-frame technique which is used in order to enforce the compensation phenomenon in the Euler-Lagrange equations [Hél91]. Finally, T. Rivière [Riv07] was able to prove regularity for critical points of general conformally invariant functionals, thus solving a conjecture by S. Hildebrandt [Hil82]. In his ingenious approach he applies a technique based on K. Uhlenbeck's results in gauge theory [Uhl82] in order to implement div-curl expressions in the Euler-Lagrange equations, a technique which can be reinterpreted as an extension of Hélein's moving frame method; see [Sch10a]. For more details and references we refer to Hélein's book [Hél02] and the extensive introduction in [Riv07] as well as [Riv09].
Naturally, it is interesting to see how these results extend to other dimensions: In the four-dimensional case, regularity can be proven for critical points of the following functional, the so-called extrinsic biharmonic maps:

$$E_4(u) := \int\limits_{B_1(0)\subset\mathbb{R}^4} |\Delta u|^2, \qquad \text{where } u \in W^{2,2}(B_1(0), \mathbb{R}^m).$$

This was done by A. Chang, L. Wang, and P. Yang [CWY99] in the case of a sphere as the target manifold, and for more general targets by P. Strzelecki [Str03], C. Wang [Wan04] and C. Scheven [Sch08]; see also T. Lamm and T. Rivière's paper [LR08]. More generally, for all *even* $n \geq 6$ similar regularity results hold, and we refer to the work of A. Gastel and C. Scheven [GS09] as well as the article of P. Goldstein, P. Strzelecki and A. Zatorska-Goldstein [GSZG09].
In *odd* dimensions non-local operators appear, and only two results for dimension $n = 1$ are available. In [DLR09], F. Da Lio and T. Rivière prove Hölder continuity for critical points of the functional

$$E_1(u) = \int\limits_{\mathbb{R}^1} \left|\Delta^{\frac{1}{4}} u\right|^2, \qquad \text{defined on distributions } u \text{ with finite energy and } u \in \mathbb{S}^{m-1} \text{ a.e.}$$

In [DLR10] this is extended to the setting of general target manifolds.

In general, we consider for $n, m \in \mathbb{N}$ and some domain $D \subset \mathbb{R}^n$ the regularity of critical points on D of the functional

$$E_n(v) = \int\limits_{\mathbb{R}^n} \left|\Delta^{\frac{n}{4}} v\right|^2, \qquad v \in H^{\frac{n}{2}}(\mathbb{R}^n, \mathbb{R}^m),\ v \in \mathbb{S}^{m-1} \text{ a.e. in } D. \tag{1.1}$$

Here, $\Delta^{\frac{n}{4}}$ denotes the operator which acts on functions $v \in L^2(\mathbb{R}^n)$ according to

$$\left(\Delta^{\frac{n}{4}} v\right)^\wedge(\xi) = |\xi|^{\frac{n}{2}}\ v^\wedge(\xi) \qquad \text{for almost every } \xi \in \mathbb{R}^n,$$

where $()^\wedge$ denotes the application of the Fourier transform. The space $H^{\frac{n}{2}}(\mathbb{R}^n)$ is the space of all functions $v \in L^2(\mathbb{R}^n)$ such that $\Delta^{\frac{n}{4}} v \in L^2(\mathbb{R}^n)$. The term "critical point" is defined as usual:

Definition 1.1 (Critical Point). *Let $u \in H^{\frac{n}{2}}(\mathbb{R}^n, \mathbb{R}^m)$, $D \subset \mathbb{R}^n$. We say that u is a critical point of $E_n(\cdot)$ on D if $u(x) \in \mathbb{S}^{m-1}$ for almost every $x \in D$ and*

$$\left.\frac{d}{dt}\right|_{t=0} E(u_{t,\varphi}) = 0$$

for any $\varphi \in C_0^\infty(D, \mathbb{R}^m)$ where $u_{t,\varphi} \in H^{\frac{n}{2}}(\mathbb{R}^n)$ is defined as

$$u_{t,\varphi} = \begin{cases} \Pi(u + t\varphi) & \text{in } D, \\ u & \text{in } \mathbb{R}^n \backslash D. \end{cases}$$

Here, Π *denotes the orthogonal projection from a tubular neighborhood of* $\mathbb{S}^{m-1}$ *into* $\mathbb{S}^{m-1}$ *defined as* $\Pi(x) = \frac{x}{|x|}$.

If n is an even number, the domain of $E_n(\cdot)$ is just the classic Sobolev space $H^{\frac{n}{2}}(\mathbb{R}^n) \equiv W^{\frac{n}{2},2}(\mathbb{R}^n)$, for odd dimensions this is a fractional Sobolev space (see Section 2.2). Functions in $H^{\frac{n}{2}}(\mathbb{R}^n)$ can contain logarithmic singularities (cf. [Fre73]) but this space embeds continuously into $BMO(\mathbb{R}^n)$, and even only slightly improved integrability or more differentiability would imply continuity.
In the light of the existing results in even dimensions and in the one-dimensional case, one may expect that similar regularity results should hold for any dimension. As a first step in that direction, we establish regularity of $n/2$-harmonic maps into the sphere.

Theorem 1.2. *For any* $n \geq 1$*, critical points* $u \in H^{\frac{n}{2}}(\mathbb{R}^n)$ *of* E_n *on a domain* D *are locally Hölder continuous in* D*.*

Note that here – in contrast to [DLR09] – we work on general domains $D \subseteq \mathbb{R}^n$. This is motivated by the facts that Hölder continuity is a local property and that $\Delta^{\frac{n}{4}}$ (though it is a non-local operator) still behaves "pseudo-local". Thus, we can impose our conditions (here: being a critical point and mapping into the sphere) only in some domain $D \subset \mathbb{R}^n$, and still get interior regularity within D.
Let us comment on the strategy of the proof. As said before, in all even dimensions the key tool for proving regularity is the discovery of compensation phenomena built into the respective Euler-Lagrange equation. For example, critical points $u \in W^{1,2}(D, \mathbb{S}^{m-1})$ of E_2 satisfy the following Euler-Lagrange equation [Hél90]

$$\Delta u^i = u^i |\nabla u|^2, \qquad \text{weakly in } D, \quad \text{for all } i = 1 \dots m. \tag{1.2}$$

For mappings $u \in W^{1,2}(\mathbb{R}^2, \mathbb{S}^{m-1})$ this is a critical equation, as the right-hand side seems to lie only in L^1: If we had no additional information, it would seem as if the equation admitted a logarithmic singularity (for examples see, e.g., [Riv07], [Fre73]). But, using the constraint $|u| \equiv 1$, one can rewrite the right-hand side of (1.2) as

$$u^i |\nabla u|^2 = \sum_{j=1}^{m} \left(u^i \nabla u^j - u^j \nabla u^i\right) \cdot \nabla u^j = \sum_{j=1}^{m} \left(\partial_1 B_{ij}\ \partial_2 u^j - \partial_2 B_{ij}\ \partial_1 u^j\right)$$

where the B_{ij} are chosen such that $\partial_1 B_{ij} = u^i \partial_2 u^j - u^j \partial_2 u^i$, and $-\partial_2 B_{ij} = u^i \partial_1 u^j - u^j \partial_1 u^i$, a choice which is possible due to Poincaré's Lemma and because (1.2) implies $\operatorname{div}\left(u^i \nabla u^j - u^j \nabla u^i\right) = 0$ for every $i, j = 1 \dots m$. Thus, (1.2) transforms into

$$\Delta u^i = \sum_{j=1}^{m} \left(\partial_1 B_{ij}\ \partial_2 u^j - \partial_2 B_{ij}\ \partial_1 u^j\right), \tag{1.3}$$

a form whose right-hand side exhibits a compensation phenomenon which in a similar way already appeared in the so-called Wente inequality [Wen69], see also [BC84], [Tar85]. In fact, the right-hand side belongs to the Hardy space (cf. [Mül90], [CLMS93]) which is a proper subspace of L^1 with enhanced potential theoretic properties. Namely, members of the Hardy space behave well with Calderón-Zygmund operators, and by this one can conclude continuity of u.
An alternative and for our purpose more viable way to describe this can be found in L. Tartar's proof [Tar85] of Wente's inequality: Assume we have for $a, b \in L^2(\mathbb{R}^2)$ a solution $w \in H^1(\mathbb{R}^2)$ of

$$\Delta w = \partial_1 a\ \partial_2 b - \partial_2 a\ \partial_1 b \qquad \text{weakly in } \mathbb{R}^2. \tag{1.4}$$

Taking the Fourier-Transform on both sides, this is (formally) equivalent to

$$|\xi|^2 w^\wedge(\xi) = c \int_{\mathbb{R}^2} a^\wedge(x)\ b^\wedge(\xi - x)\left(x_1(\xi_2 - x_2) - x_2(\xi_1 - x_1)\right)\ dx, \qquad \text{for } \xi \in \mathbb{R}^2. \tag{1.5}$$

Now the compensation phenomena responsible for the higher regularity of w can be identified with the following inequality:

$$|x_1(\xi_2 - x_2) - x_2(\xi_1 - x_1)| \leq |\xi| |x|^{\frac{1}{2}} |\xi - x|^{\frac{1}{2}}. \tag{1.6}$$

Observe, that $|x|$ as well as $|\xi - x|$ appear to the power $1/2$, only. Interpreting these factors as Fourier multipliers, this means that only "half of the gradient", more precisely $\Delta^{\frac{1}{4}}$, of a and b enters the equation, which implies that the right-hand side is a "product of lower order" operators. In fact, plugging (1.6) into (1.5), one can conclude $w^\wedge \in L^1(\mathbb{R}^2)$ just by Hölder's and Young's inequality on Lorentz spaces – consequently one has proven continuity of w, because the inverse Fourier transform maps L^1 into C^0. As explained earlier, (1.2) can be rewritten as (1.3) which has the form of (1.4), thus we have continuity for critical points of E_2, and by a bootstraping argument (see [Tom69]) one gets analyticity of these points.

As in Theorem 1.2 we prove only interior regularity, it is natural to work with localized Euler-Lagrange equations which look as follows, see Section 7:

Lemma 1.3 (Euler-Lagrange Equations). *Let $u \in H^{\frac{n}{2}}(\mathbb{R}^n)$ be a critical point of E_n on a domain $D \subset \mathbb{R}^n$. Then, for any cutoff function $\eta \in C_0^\infty(D)$, $\eta \equiv 1$ on an open neighborhood of a ball $\tilde{D} \subset D$ and $w := \eta u$, we have*

$$- \int_{\mathbb{R}^n} w^i \ \Delta^{\frac{n}{4}} w^j \ \Delta^{\frac{n}{4}} \psi_{ij} = \int_{\mathbb{R}^n} \Delta^{\frac{n}{4}} w^j \ H(w^i, \psi_{ij}) - \int_{\mathbb{R}^n} a_{ij} \psi_{ij}, \quad \text{for any } \psi_{ij} = -\psi_{ji} \in C_0^\infty(\tilde{D}), \tag{1.7}$$

where $a_{ij} \in L^2(\mathbb{R}^n)$, $i,j = 1, \dots, m$, depend on the choice of η. Here, we adopt Einstein's summation convention. Moreover, $H(\cdot,\cdot)$ is defined on $H^{\frac{n}{2}}(\mathbb{R}^n) \times H^{\frac{n}{2}}(\mathbb{R}^n)$ as

$$H(a,b) := \Delta^{\frac{n}{4}}(ab) - a \Delta^{\frac{n}{4}} b - b \Delta^{\frac{n}{4}} a, \qquad \text{for } a, b \in H^{\frac{n}{2}}(\mathbb{R}^n). \tag{1.8}$$

Furthermore, $u \in \mathbb{S}^{m-1}$ on D implies the following structure equation

$$w^i \cdot \Delta^{\frac{n}{4}} w^i = -\frac{1}{2} H(w^i, w^i) + \frac{1}{2} \Delta^{\frac{n}{4}} \eta^2 \qquad \text{a.e. in } \mathbb{R}^n. \tag{1.9}$$

Similar in its spirit to [DLR09] we use that (1.7) and (1.9) together control the full growth of $\Delta^{\frac{n}{4}} w$, though here we use a different argument applying an analogue of Hodge decomposition to show this, see below. Note moreover that as we have localized our Euler-Lagrange equation, we do not need further rewriting of the structure condition (1.9) as was done in [DLR09].
While in (1.4) the compensation phenomenon stems from the structure of the right-hand side, here it comes from the leading order term $H(\cdot,\cdot)$ appearing in (1.7) and (1.9). This can be proved by Tartar's approach [Tar85], using essentially only the following elementary "compensation inequality" similar in its spirit to (1.6)

$$\left| |x - \xi|^p - |\xi|^p - |x|^p \right| \leq C_p \begin{cases} |x|^{p-1}|\xi| + |\xi|^{p-1}|x|, & \text{if } p > 1, \\ |x|^{\frac{p}{2}} |\xi|^{\frac{p}{2}}, & \text{if } p \in (0,1]. \end{cases} \tag{1.10}$$

More precisely, we will prove in Section 4

Theorem 1.4. *For H as in (1.8) and $u, v \in H^{\frac{n}{2}}(\mathbb{R}^n)$ one has*

$$\|H(u,v)\|_{L^2(\mathbb{R}^n)} \leq C \ \|(\Delta^{\frac{n}{4}} u)^\wedge\|_{L^2(\mathbb{R}^n)} \ \|(\Delta^{\frac{n}{4}} v)^\wedge\|_{L^{2,\infty}(\mathbb{R}^n)}.$$

An equivalent compensation phenomenon was observed in the case $n = 1$ in [DLR09][1]. Note that interpreting again the terms of (1.10) as Fourier multipliers, it seems as if this equation (and as a consequence Theorem 1.4) estimates the operator $H(u,v)$ by products of lower order operators applied to u and v. Here, by "products of lower order operators" we mean products of operators whose differential order is strictly between zero and $\frac{n}{2}$ and where the two operators together give an operator of order $\frac{n}{2}$. In fact, this is exactly what happens in special cases, e.g. $H(u,v) = 2\nabla u \cdot \nabla u$ if we take the case $n = 4$ where $\Delta^{\frac{n}{4}} = \Delta$.
Another case we will need to control is the case where $u = P$ is a polynomial of degree less than $\frac{n}{2}$. As (at least formally) $\Delta^{\frac{n}{4}} P = 0$ this is to estimate

$$H(P,v) = \Delta^{\frac{n}{4}}(Pv) - P \Delta^{\frac{n}{4}} v.$$

This case is not contained in Theorem 1.4 as a non-zero polynomial does not belong to $H^{\frac{n}{2}}(\mathbb{R}^n)$. Obviously, in the one-dimensional case P is only a constant, and thus $H(P,v) \equiv 0$. In higher dimensions, this term does not vanish. However, we will show in Proposition 5.12 that $H(P,v)$ is still a product of lower order expressions.
As we are going to show in Section 5.4, products of lower order operators (in the way this term is defined above) "localize well". By that we mean that the L^2-norm of such a product evaluated on a ball is estimated by the product of L^2-norms of $\Delta^{\frac{n}{4}}$ applied to the factors evaluated at a slightly bigger ball, up to harmless error terms. As a consequence, one expects this to hold as well for the term $H(u,v)$, and in fact, we can show the following "localized version" of Theorem 1.4, proven in Section 6.

[1] In fact, all compensation phenomena established in [DLR09] can be proven by our adaptation of Tartar's method using simple compensation inequalities, thus avoiding the use of paraproduct arguments (but at the expense of using the theory of Lorentz spaces).

Theorem 1.5 (Localized Compensation Results)**.** *There is a uniform constant $\gamma > 0$ depending only on the dimension n, such that the following holds. Let $H(\cdot,\cdot)$ be defined as in (1.8). For any $v \in H^{\frac{n}{2}}(\mathbb{R}^n)$ and $\varepsilon > 0$ there exist constants $R > 0$ and $\Lambda_1 > 0$ such that for any ball $B_r(x) \subset \mathbb{R}^n$, $r \in (0,R)$,*

$$\|H(v,\varphi)\|_{L^2(B_r(x))} \leq \varepsilon\ \|\Delta^{\frac{n}{4}}\varphi\|_{L^2(\mathbb{R}^n)} \qquad \text{for any } \varphi \in C_0^\infty(B_r(x)),$$

and

$$\|H(v,v)\|_{L^2(B_r(x))} \leq \varepsilon\ [[v]]_{B_{\Lambda_1 r}(x)} + C_{\varepsilon,v} \sum_{k=-\infty}^{\infty} 2^{-\gamma|k|}[[v]]_{B_{2^{k+1}r}(x)\backslash B_{2^k r}(x)}.$$

Here, $[[v]]_A$ is a pseudo-norm, which in a way measures the L^2-norm of $\Delta^{\frac{n}{4}}v$ on $A \subset \mathbb{R}^n$. More precisely,

$$[[v]]_A := \|\Delta^{\frac{n}{4}}v\|_{L^2(A)} + \left(\int\limits_A \int\limits_A |x-y|^{-n-1} \left|\nabla^{\frac{n-1}{2}}v(x) - \nabla^{\frac{n-1}{2}}v(y)\right|^2 \,dx\,dy\right)^{\frac{1}{2}}, \quad \text{for } n \text{ odd},$$

and for even n we set $[[v]]_A := \|\Delta^{\frac{n}{4}}v\|_{L^2(A)} + \|\nabla^{\frac{n}{2}}v\|_{L^2(A)}$.

As mentioned before, by the structure of our Euler-Lagrage equations, these local estimates control the local growth of the $\frac{n}{4}$-operator of any critical point. This is true, as local growth of L^2-functions is controlled by their local weak $\Delta^{\frac{n}{4}}$-norm. More precisely, we will show the following result in Section 5.3 using an analogue of the Hodge decomposition, see Lemma 2.9.

Theorem 1.6. *There are uniform constants $\Lambda_2 > 0$ and $C > 0$ such that the following holds: For any $x \in \mathbb{R}^n$ and any $r > 0$ we have for every $v \in L^2(\mathbb{R}^n)$, $\operatorname{supp} v \subset B_r(x)$,*

$$\|v\|_{L^2(B_r(x))} \leq C \sup_{\varphi \in C_0^\infty(B_{\Lambda_2 r}(x))} \frac{1}{\|\Delta^{\frac{n}{4}}\varphi\|_{L^2(\mathbb{R}^n)}} \int\limits_{\mathbb{R}^n} v\ \Delta^{\frac{n}{4}}\varphi.$$

Then, by an iteration technique adapted from the one in [DLR09] (see the appendix) we conclude in Section 9 that the critical point u of E_n lies in a Morrey-Campanato space, which implies Hölder continuity.
As for the sections not mentioned so far: In Section 2 we will cover some basic facts on Lorentz and Sobolev spaces. In Section 3 we will prove a fractional Poincaré inequality with a mean value condition of arbitrary order. In Section 5 various localizing effects are studied. In Section 8 we compare two pseudo-norms $\|\Delta^{\frac{n}{4}}v\|_{L^2(A)}$ and $[v]_{\frac{n}{2},A}$ of $H^{\frac{n}{2}}$, and finally, in Section 9, Theorem 1.2 is proved.

Finally, let us remark the following two points: As we cut off the critical points u to bounded domains, the assumption $u \in L^2(\mathbb{R}^n)$ is not necessary, one could, e.g., assume $u \in L^\infty(\mathbb{R}^n)$, $\Delta^{\frac{n}{4}}u \in L^2(\mathbb{R}^n)$, thus regaining a similar "global" result as in [DLR09]. Observe moreover, that the application of a cut-off function within D to the critical point u is a rather brute operation, which nevertheless suffices our purposes as in this note we are only interested in *interior* regularity. For the analysis of the boundary behavior of u one probably would need a more careful cut-off argument.

We will use fairly standard *notation*:
As usual, we denote by $\mathcal{S} \equiv \mathcal{S}(\mathbb{R}^n)$ the Schwartz class of all smooth functions which at infinity go faster to zero than any quotient of polynomials, and by $\mathcal{S}' \equiv \mathcal{S}'(\mathbb{R}^n)$ its dual. For a set $A \subset \mathbb{R}^n$ we will denote its n-dimensional Lebesgue measure by $|A|$, and rA, $r > 0$, will be the set of all points $rx \in \mathbb{R}^n$ where $x \in A$. By $B_r(x) \subset \mathbb{R}^n$ we denote the open ball with radius r and center $x \in \mathbb{R}^n$. If no confusion arises, we will abbreviate $B_r \equiv B_r(x)$. When we speak of a multiindex α we will usually mean $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n \equiv (\mathbb{N}_0)^n$ with length $|\alpha| := \sum_{i=1}^n \alpha_i$. For such a multiindex α and $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ we denote by $x^\alpha = \prod_{i=1}^n (x_i)^{\alpha_i}$ where we set $(x_i)^0 := 1$ even if $x_i = 0$. For a real number $p \geq 0$ we denote by $\lfloor p \rfloor$ the biggest integer below p and by $\lceil p \rceil$ the smallest integer above p. If $p \in [1,\infty]$ we usually will denote by p' the Hölder conjugate, that is $\frac{1}{p} + \frac{1}{p'} = 1$. By $f * g$ we denote the convolution of two functions f and g. As mentioned before, we will denote by $f^\wedge$ the Fourier transform and by $f^\vee$ the inverse Fourier transform, which on the Schwartz class $\mathcal{S}$ are defined as

$$f^\wedge(\xi) := \int\limits_{\mathbb{R}^n} f(x)\ e^{-2\pi \mathbf{i}\ x\cdot\xi}\ dx, \quad f^\vee(x) := \int\limits_{\mathbb{R}^n} f(\xi)\ e^{2\pi \mathbf{i}\ \xi\cdot x}\ d\xi.$$

By $\mathbf{i}$ we denote here and henceforth the imaginary unit $\mathbf{i}^2 = -1$. $\mathcal{R}$ is the Riesz operator which transforms $v \in \mathcal{S}(\mathbb{R}^n)$ according to $(\mathcal{R}v)^\wedge(\xi) := \mathbf{i}\frac{\xi}{|\xi|}v^\wedge(\xi)$. More generally, we will speak of a zero-multiplier operator M, if there is a function $m \in C^\infty(\mathbb{R}^n\backslash\{0\})$ homogeneous of order 0 and such that $(Mv)^\wedge(\xi) = m(\xi)\ v^\wedge(\xi)$ for all $\xi \in \mathbb{R}^n\backslash\{0\}$. For a measurable set $D \subset \mathbb{R}^n$, we denote the integral mean of an integrable function $v : D \to \mathbb{R}$ to be $(v)_D \equiv \fint_D v \equiv \frac{1}{|D|}\int_D v$. Lastly, our constants – usually denoted by C or c – can possibly change from line to line and usually depend on the space dimensions involved. Further dependencies will be indicated by a subscript, though we will make no effort to pin down the exact value of those constants. If we consider the constant factors to be irrelevant with respect to the mathematical argument, for the sake of simplicity we will omit them in the calculations, writing $\prec$, $\succ$, $\approx$ instead of $\leq$, $\geq$ and $=$.

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2 Lorentz-, Sobolev Spaces and Cutoff Functions

2.1 Lorentz Spaces

In this section, we recall the definition of Lorentz spaces, which are a refinement of the standard Lebesgue-spaces. For more on Lorentz spaces, the interested reader might consider [Hun66], [Zie89], [Gra08, Section 1.4], as well as [Tar07].

Definition 2.1 (Lorentz Space)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be measurable and set $d_f(\lambda) := |\{x \in \mathbb{R}^n\ :\ |f(x)| > \lambda\}|$. The decreasing rearrangement of f is the function f^* defined on $[0,\infty)$ by $f^*(t) := \inf\{s > 0\ :\ d_f(s) \leq t\}$. For $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, the Lorentz space $L^{p,q} \equiv L^{p,q}(\mathbb{R}^n)$, is the set of measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $\|f\|_{L^{p,q}} < \infty$, where*

$$\|f\|_{L^{p,q}} := \begin{cases} \left(\int\limits_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}, & \text{if } q < \infty,\\ \sup_{t>0} t^{\frac{1}{p}} f^*(t), & \text{if } q = \infty,\ p < \infty,\\ \|f\|_{L^\infty(\mathbb{R}^n)}, & \text{if } q = \infty,\ p = \infty. \end{cases}$$

Observe that $\|\cdot\|_{L^{p,q}}$ does not *satisfy the triangle inequality.*

There is another definition of Lorentz spaces by interpolation between L^1 and L^p, cf. [Tar07]. Note that we have not defined the space $L^{\infty,q}$ for $q \in [1,\infty)$. For simplicity, whenever a result on Lorentz spaces is stated in a way that $L^{p,q}$ for $p = \infty$, $q \in [1,\infty]$ is admissible, we in fact only claim that result for $p = \infty$, $q = \infty$. Next, we state some basic properties of Lorentz spaces. The proofs are either easy exercises or they are contained in the above mentioned articles and monographs (cf. also [Sch10b]).

Proposition 2.2. *Let $f \in L^{p_1,q_1}$ and $g \in L^{p_2,q_2}$, $1 \leq p_1, p_2, q_1, q_2 \leq \infty$.*

(i) If $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} \in [0,1]$ and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$ then $fg \in L^{p,q}$ and $\|fg\|_{L^{p,q}} \prec \|f\|_{L^{p_1,q_1}}\ \|g\|_{L^{p_2,q_2}}$.

(ii) If $\frac{1}{p_1} + \frac{1}{p_2} - 1 = \frac{1}{p}$, $1 < p_1, p_2, p < \infty$, and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$ then $f * g \in L^{p,q}$ and $\|f * g\|_{L^{p,q}} \prec \|f\|_{L^{p_1,q_1}}\ \|g\|_{L^{p_2,q_2}}$.

(iii) For $p_1 \in (1,\infty)$, f belongs to $L^{p_1}(\mathbb{R}^n)$ if and only if $f \in L^{p_1,p_1}$. The "norms" of L^{p_1,p_1} and L^{p_1} are equivalent.

(iv) If $p_1 \in (1,\infty)$ and $q \in [q_1,\infty]$ then also $f \in L^{p_1,q}$.

(v) $\frac{1}{|\cdot|^\lambda} \in L^{\frac{n}{\lambda},\infty}$, whenever $\lambda \in (0,n)$.

(vi) If $p_1 \in (1,2)$, $q_1 \in [1,\infty]$ we have $\|f^\wedge\|_{L^{p_1',q_1}} \leq C_{p_1}\|f\|_{L^{p_1,q_1}}$ and $\|f^\vee\|_{L^{p_1',q_1}} \leq C_{p_1}\|f\|_{L^{p_1,q_1}}$.

(vii) Let $\lambda > 0$. If we denote $\tilde{f}(\cdot) := f(\lambda\cdot)$, then $\|\tilde{f}\|_{L^{p_1,q_1}} = \lambda^{-\frac{n}{p_1}}\|f\|_{L^{p_1,q_1}}$.

(viii) Let $\operatorname{supp} f \subset \bar{D}$, where $D \subset \mathbb{R}^n$ is a bounded measurable set. Then, whenever $\infty > p_1 > p \geq 1$, $q \in [1,\infty]$, we have $\|f\|_{L^{p,q}} \leq C_{p,p_1,q}\ |D|^{\frac{1}{p}-\frac{1}{p_1}}\ \|f\|_{L^{p_1}}$

2.2 Fractional Sobolev Spaces

Definition 2.3 (Fractional Sobolev Spaces by Fourier Transform). *Let $f \in L^2(\mathbb{R}^n)$. We say that for some $s \geq 0$ the function $f \in H^s \equiv H^s(\mathbb{R}^n)$ if and only if $\Delta^{\frac{s}{2}} f \in L^2(\mathbb{R}^n)$. Here, the operator $\Delta^{\frac{s}{2}}$ is defined as $\Delta^{\frac{s}{2}} f := (|\cdot|^s f^\wedge)^\vee$. The norm, under which $H^s(\mathbb{R}^n)$ becomes a Hilbert space is $\|f\|^2_{H^s(\mathbb{R}^n)} := \|f\|^2_{L^2(\mathbb{R}^n)} + \|\Delta^{\frac{s}{2}} f\|^2_{L^2(\mathbb{R}^n)}$.*

In Section 2.3 we will state an integral representation for the fractional laplacian $\Delta^{\frac{s}{2}}$. Observe, that the definition of $\Delta^{\frac{2}{2}}$ coincides with the usual laplacian only up to a multiplicative constant, but this saves us from the nuisance to deal with those standard factors in every single calculation.
Our next goal is Poincaré's inequality. As we want to use the standard blow up argument to prove it, we premise a (trivial) uniqueness and a compactness result:

Lemma 2.4 (Uniqueness of solutions). *Let $f \in H^s(\mathbb{R}^n)$, $s > 0$. If $\Delta^{\frac{s}{2}} f \equiv 0$, then $f \equiv 0$.*

Lemma 2.5 (Compactness). *Let $D \subset \mathbb{R}^n$ be a smoothly bounded domain, $s > 0$. Assume that there is a constant $C > 0$ and $f_k \in H^s(\mathbb{R}^n)$, $k \in \mathbb{N}$, such that for any $k \in \mathbb{N}$ the conditions $\operatorname{supp} f_k \subset \bar{D}$ and $\|f_k\|_{H^s} \leq C$ hold. Then there exists a subsequence f_{k_i}, such that $f_{k_i} \xrightarrow{i \to \infty} f \in H^s$ weakly in H^s, strongly in $L^2(\mathbb{R}^n)$, and pointwise almost everywhere. Moreover, $\operatorname{supp} f \subset \bar{D}$.*

Proof of Lemma 2.5.
Fix $D \subset \mathbb{R}^n$ and let $\eta \in C_0^\infty(2D)$, $\eta \equiv 1$ on D. One shows that the operator $S: \ v \mapsto \eta v$ is compact as an operator $H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, by interpolation [Tar07, Lemma 41.4] and the fact that it is compact for any $s \in \mathbb{N}$.

Lemma 2.5 □

With the compactness lemma, Lemma 2.5, at hand we can prove the following Poincaré inequality by the usual blowup proof (for details see [Sch10b]). As in [DLR09, Theorem A.2] we will use a support-condition in order to ensure compactness of the embedding $H^s(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. This support condition can be seen as saying that all derivatives up to order $\lfloor \frac{s}{2} \rfloor$ are zero at the boundary, therefore it is not surprising that such an inequality should hold.

Lemma 2.6 (Poincaré Inequality). *For any smoothly bounded domain $D \subset \mathbb{R}^n$, $s > 0$, there exists a constant $C_{D,s} > 0$ such that*

$$\|f\|_{L^2(\mathbb{R}^n)} \leq C_{D,s} \, \|\Delta^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^n)}, \quad \text{for all } f \in H^s(\mathbb{R}^n), \operatorname{supp} f \subset \bar{D}. \tag{2.1}$$

If $D = r\tilde{D}$ for some $r > 0$, then $C_{D,s} = C_{\tilde{D},s} r^s$.

One checks as well, that $C_{D,s} = C_{\tilde{D},s}$ if D is a mere translation of some smoothly bounded domain $\tilde{D}$. This is clear, as the operator $\Delta^{\frac{s}{2}}$ commutes with translations.
A simple consequence of the "standard Poincaré inequality" is the following

Lemma 2.7 (Slightly more general Poincaré inequality). *For any smoothly bounded domain $D \subset \mathbb{R}^n$, $0 < s \leq t$, there exists a constant $C_{D,t} > 0$ such that*

$$\|\Delta^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^n)} \leq C_{D,t} \, \|\Delta^{\frac{t}{2}} f\|_{L^2(\mathbb{R}^n)}, \quad \text{for all } f \in H^t(\mathbb{R}^n), \operatorname{supp} f \subset \bar{D}.$$

If $D = r\tilde{D}$ for some $r > 0$, then $C_{D,t} = C_{\tilde{D},t} r^{t-s}$.

Proof of Lemma 2.7.
This follows by the following estimate and scaling:

$$\|\Delta^{\frac{s}{2}} f\|_{L^2} = \||\cdot|^s \ f^\wedge\|_{L^2} \leq \||\cdot|^t \ f^\wedge\|_{L^2(\mathbb{R}^n \setminus B_1(0))} + \|f^\wedge\|_{L^2(B_1(0))} \leq \|\Delta^{\frac{t}{2}} f\|_{L^2} + \|f\|_{L^2} \overset{L.2.6}{\leq} C_{D,t} \, \|\Delta^{\frac{t}{2}} f\|_{L^2}.$$

Lemma 2.7 □

The following lemma can be interpreted as an existence result for the equation $\Delta^{\frac{s}{2}} w = v$ - or as a variant of Poincaré's inequality:

Lemma 2.8. *Let $s \in (0, n)$, $p \in [2, \infty)$ such that*

$$\frac{n-s}{n} > \frac{1}{p} \geq \frac{n-2s}{2n}. \tag{2.2}$$

Then for any smoothly bounded set $D \subset \mathbb{R}^n$ there is a constant $C_{D,s,p}$ such that for any $v \in \mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp} v \subset \bar{D}$, we have $\Delta^{-\frac{s}{2}} v \in L^p(\mathbb{R}^n)$ and

$$\|\Delta^{-\frac{s}{2}} v\|_{L^p(\mathbb{R}^n)} \leq C_{D,p,s}\ \|v\|_{L^2}.$$

Here, $\Delta^{-\frac{s}{2}} v$ is defined as $(|\cdot|^{-s} v^\wedge)^\vee$. In particular, if $s \in (0, \frac{n}{2})$,

$$\|\Delta^{-\frac{s}{2}} v\|_{L^2(\mathbb{R}^n)} \leq C_{D,s}\ \|v\|_{L^2}.$$

If $D = r\tilde{D}$, then $C_{D,p,s} = r^{s+\frac{n}{p}-\frac{n}{2}}\ C_{\tilde{D},p,s}$.

***Proof of Lemma* 2.8.**
We want to make the following reasoning rigorous:

$$\|\Delta^{-\frac{s}{2}} v\|_{L^p} \overset{\substack{P.2.2 \\ p\in[2,\infty)}}{\leq} C_p\ \||\cdot|^{-s}\ v^\wedge\|_{L^{p',p}} \overset{(\star)}{\leq} C_p\ \||\cdot|^{-s}\|_{L^{\frac{n}{s},\infty}}\ \|v^\wedge\|_{L^{q,p}} \overset{\substack{P.2.2 \\ q\geq 2}}{\leq} C_{p,s,q}\ \|v\|_{L^{q',2}} \overset{\substack{P.2.2 \\ q'\leq 2}}{\leq} C_{s,q}\ C_D\ \|v\|_{L^2}.$$

To do so, we need to find $q \in [2, \infty)$ such that $(\star)$ holds $\frac{1}{p'} = \frac{1}{q} + \frac{s}{n}$, which is possible by virtue of (2.2). Then the validity of $(\star)$ follows from Proposition 2.2 and we conclude with scaling.

Lemma 2.8 □

The next lemma can be interpreted as an adaption of Hodge decomposition to the setting of the fractional laplacian:

Lemma 2.9 (Hodge Decomposition)**.** *Let $f \in L^2(\mathbb{R}^n)$, $s > 0$. Then for any smoothly bounded domain $D \subset \mathbb{R}^n$ there are functions $\varphi \in H^s(\mathbb{R}^n)$, $\operatorname{supp} \varphi \subset \bar{D}$, and $h \in L^2(\mathbb{R}^n)$ such that $f = \Delta^{\frac{s}{2}} \varphi + h$ almost everywhere in $\mathbb{R}^n$ and*

$$\int\limits_{\mathbb{R}^n} h\ \Delta^{\frac{s}{2}} \psi = 0, \quad \text{for all } \psi \in C_0^\infty(D).$$

Moreover, $\|h\|_{L^2(\mathbb{R}^n)} + \|\Delta^{\frac{s}{2}} \varphi\|_{L^2(\mathbb{R}^n)} \leq 5 \|f\|_{L^2(\mathbb{R}^n)}$.

***Proof of Lemma* 2.9.**
Set

$$E(v) := \int\limits_{\mathbb{R}^n} \left|\Delta^{\frac{s}{2}} v - f\right|^2, \quad \text{for } v \in H^s(\mathbb{R}^n) \text{ with } \operatorname{supp} v \subset \bar{D}.$$

One can prove via Poincaré's inequality, Lemma 2.6, and the compactness lemma, Lemma 2.5, that E is coercive and that consequently there exists a minimizer φ of $E(\cdot)$ in $H^s(\mathbb{R}^n)$ with the support condition $\operatorname{supp} \varphi \subset D$. If we call $h := \Delta^{\frac{s}{2}} \varphi - f$, Euler-Lagrange-Equations and the minimization process itself imply the claimed properties.

Lemma 2.9 □

In fact, h will satisfy enhanced local estimates, similar to estimates for harmonic function, see Lemma 5.8.

2.3 An Integral Definition for the Fractional Laplacian

A further definition of the fractional laplacian for small order without the use of the Fourier transform are based on the following proposition.

Proposition 2.10 (Fractional Laplacian - Integral Definition)**.** *(i) Let $s \in (0, 1)$. For some constant c_n and any $v \in \mathcal{S}(\mathbb{R}^n)$,*

$$\Delta^{\frac{s}{2}} v(\bar{y}) = c_n \int\limits_{\mathbb{R}^n} \frac{v(x) - v(\bar{y})}{|x - \bar{y}|^{n+s}}\ dx \quad \text{for any } \bar{y} \in \mathbb{R}^n.$$

(ii) Let $s \in (0, 2)$. Then,

$$\Delta^{\frac{s}{2}} v(\bar{y}) = \frac{1}{2} c_n \int\limits_{\mathbb{R}^n} \frac{v(\bar{y} - x) + v(\bar{y} + x) - 2v(\bar{y})}{|x|^{n+s}}\ dx.$$

(iii) For any $s \in (0, 2)$, $v, w \in \mathcal{S}(\mathbb{R}^n)$

$$\int\limits_{\mathbb{R}^n} \Delta^{\frac{s}{2}} v\ w = c_n \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \frac{(v(x) - v(y))\ (w(y) - w(x))}{|x - y|^{n+s}}\ dx\ dy.$$

(iv) Let $s \in (0,1)$. For a constant $c_n > 0$ and for any $v \in \mathcal{S}(\mathbb{R}^n)$

$$\|\Delta^{\frac{s}{2}} v\|^2_{L^2(\mathbb{R}^n)} = c_n \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\, dx\, dy.$$

We will introduce the pseudo-norm $[v]_{D,s}$, a quantity which for $s \in (0,1)$ actually is equivalent to the local, homogeneous H^s-norm, see [Tar07], [Tay96]. But we will not use this fact as we will work with $s = \frac{n}{2}$ for $n \in \mathbb{N}$, including $n \in \mathbb{N}$ greater than 2. Nevertheless, we will see in Section 8 that $[v]_{D,\frac{n}{2}}$ is "almost" comparable to $\|\Delta^{\frac{n}{4}} v\|_{L^2(D)}$.

Definition 2.11. *For a domain $D \subset \mathbb{R}^n$ and $s \geq 0$ we set*

$$([u]_{D,s})^2 := \int\limits_D \int\limits_D \frac{\left|\nabla^{\lfloor s \rfloor} u(z_1) - \nabla^{\lfloor s \rfloor} u(z_2)\right|^2}{|z_1 - z_2|^{n+2(s-\lfloor s \rfloor)}}\, dz_1\, dz_2 \tag{2.3}$$

if $s \notin \mathbb{N}_0$. If $s \in \mathbb{N}_0$ we just define $[u]_{D,s} = \|\nabla^s u\|_{L^2(D)}$.

Observe that by the definition of $[\cdot]_{D,s}$ it is obvious that for any polynomial P of degree less than s,

$$[v+P]_{D,s} = [v]_{D,s}.$$

2.4 Annuli-Cutoff Functions

We will have to localize our equations, so we introduce as in [DLR09] a decomposition of unity as follows: Let $\eta \equiv \eta^0 \in C_0^\infty(B_2(0))$, $\eta \equiv 1$ in $B_1(0)$ and $0 \leq \eta \leq 1$ in $\mathbb{R}^n$. Let furthermore $\eta^k \in C_0^\infty(B_{2^{k+1}}(0) \backslash B_{2^{k-1}}(0))$, $k \in \mathbb{N}$, such that $0 \leq \eta^k \leq 1$, $\sum_{k=0}^\infty \eta^k = 1$ pointwise in $\mathbb{R}^n$ and $\left|\nabla^i \eta^k\right| \leq C_i 2^{-ki}$ for any $i \in \mathbb{N}_0$. We call $\eta^k_{r,x} := \eta^k(\frac{\cdot - x}{r})$, though we will often omit the subscript when x and r should be clear from the context.
We want to estimate some L^p-Norms of $\Delta^{\frac{s}{2}} \eta^k_{r,x}$. In order to do so, we will need the following Proposition which can be proven similar to [Gra08, Exercise 2.2.14, p.108].

Proposition 2.12. *For every $g \in \mathcal{S}(\mathbb{R}^n)$, $p \in [1,2]$, $s \geq 0$, $-\infty < \alpha < n\frac{p-2}{p} < \beta < \infty$, we have*

$$\|\left(\Delta^{\frac{s}{2}} g\right)^\wedge\|_{L^p(\mathbb{R}^n)} \leq C_{\alpha,\beta,p} \left(\|\Delta^{\frac{s+\alpha}{2}} g\|_{L^2(\mathbb{R}^n)} + \|\Delta^{\frac{s+\beta}{2}} g\|_{L^2(\mathbb{R}^n)}\right).$$

Proposition 2.13. *For any $s > 0$, $p \in [1,2]$, there is a constant $C_{s,p} > 0$, such that for any $k \in \mathbb{N}_0$, $x \in \mathbb{R}^n$, $r > 0$ denoting as usual $p' := \frac{p}{p-1}$,*

$$\|\left(\Delta^{\frac{s}{2}} \eta^k_{r,x}\right)^\wedge\|_{L^p(\mathbb{R}^n)} \leq C_{s,p}\ (2^k r)^{-s+\frac{n}{p'}}. \tag{2.4}$$

In particular,

$$\|\Delta^{\frac{s}{2}} \eta^k_{r,x}\|_{L^{p'}(\mathbb{R}^n)} \leq C_{s,p}\ (2^k r)^{-s+\frac{n}{p'}}. \tag{2.5}$$

Proof of Proposition 2.13.
Fix $r > 0$, $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$. Set $\tilde{\eta}(\cdot) := \eta^k_{r,x}(x + 2^k r \cdot)$. By scaling it then suffices to show that for a uniform constant $C_{s,p} > 0$

$$\|\left(\Delta^{\frac{s}{2}} \tilde{\eta}\right)^\wedge\|_{L^p(\mathbb{R}^n)} \leq C_{s,p}. \tag{2.6}$$

Scaling back we conclude the proof of (2.4). Equation (2.5) then follows by the continuity of the inverse Fourier-transform from L^p to $L^{p'}$ whenever $p \in [1,2]$, see Proposition 2.2.

Proposition 2.13 □

Remark 2.14. *One can show, that*

$$\|\Delta^{\frac{s}{2}} (\eta_{r,0} x^\alpha)\|_{L^p(\mathbb{R}^n)} \leq C_{s,p}\ r^{-s+|\alpha|+\frac{n}{p}} \quad \text{for any } p \in [2,\infty],\ |\alpha| < s,\ r > 0.$$

This is done similar to the proof of Proposition 2.13: First one proves the claim for $r = 1$, then scaling implies the claim, using that

$$\eta_{r,0}(x) x^\alpha = r^{|\alpha|} \eta_{1,0}(r^{-1}x)(r^{-1}x)^\alpha.$$

As a consequence, $\Delta^{\frac{s}{2}} P$ vanishes for a polynomial P, if s is greater than the degree of P - in a weak sense:

Proposition 2.15. *Let α be a multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$, where $\alpha_i \in \mathbb{N}_0$, $1 \le i \le n$. If $s > 0$ such that $|\alpha| = \sum_{i=1}^{n} |\alpha_i| < s$ then*

$$\lim_{R\to\infty} \int_{\mathbb{R}^n} \eta_R x^\alpha \ \Delta^{\frac{s}{2}} \varphi = 0, \quad \text{for every } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Here, $x^\alpha := (x_1)^{\alpha_1} \cdots (x_n)^{\alpha_n}$.

We will use Proposition 2.15 in a formal way, by saying that formally $\Delta^{\frac{s}{2}} x^\alpha = 0$ whenever $|\alpha| < s$. Of course, as we defined the operator $\Delta^{\frac{s}{2}}$ on L^2-Functions only, this formal argument should be verified in each calculation by using that $\lim_{R\to\infty} \Delta^{\frac{s}{2}}(\eta_R x^\alpha) = 0$, where the limit will be taken in an appropriate sense. For the sake of simplicity, now and then we will omit this recurring argument.

3 Mean Value Poincaré Inequality of Fractional Order

By the Fundamental Theorem of Calculus one can prove the following

Proposition 3.1 (Estimate on Convex Sets)**.** *Let D be a convex, bounded domain and $\gamma < n+2$, then for any $v \in C^\infty(\mathbb{R}^n)$,*

$$\int_D \int_D \frac{|v(x)-v(y)|^2}{|x-y|^\gamma} \, dx \, dy \le C_{D,\gamma} \int_D |\nabla v(z)|^2 \, dz.$$

If $\gamma = 0$, the constant $C_{D,\gamma} = C_n \ |D| \operatorname{diam}(D)^2$.

An immediate consequence for $\gamma = 0$ is the classic Poincaré inequality for mean values on convex domains.

Lemma 3.2. *There is a uniform constant $C > 0$ such that for any $v \in C^\infty(\mathbb{R}^n)$ and for any convex, bounded set $D \subset \mathbb{R}^n$*

$$\int_D |v - (v)_D|^2 \le C \ (\operatorname{diam}(D))^2 \ \|\nabla v\|^2_{L^2(D)}.$$

In the following two sections we prove in Lemma 3.5 and Lemma 3.6 higher (fractional) order analogues of this Mean-Value-Poincaré-Inequality, on the ball and on the annulus, respectively. More precisely, for η_r^k from Section 2.4 we will only show that

$$\|\Delta^{\frac{s}{2}}(\eta_r^k v)\|_{L^2(\mathbb{R}^n)} \prec \|\Delta^{\frac{s}{2}} v\|_{L^2(\mathbb{R}^n)},$$

if v satisfies a mean value condition, similar to the following: For some $N \in \mathbb{N}_0$ and a domain $D \subset \mathbb{R}^n$ (in our example e.g. $D = \operatorname{supp} \eta_r^k$ and $N = \lceil s \rceil - 1$)

$$⨍_D \partial^\alpha v = 0, \quad \text{for any multiindex } \alpha \in (\mathbb{N}_0)^n, \, |\alpha| \le N. \tag{3.1}$$

The necessary ingredients can be paraphrased as follows: For any $s > 1$ we can decompose $\Delta^{\frac{s}{2}}$ into $\Delta^{\frac{t}{2}} \circ T$ for some $t \in (0,1)$ and where T is a classic differential operator possibly plugged behind a Riesz-transform. So, we first focus in Proposition 3.4 on the case $\Delta^{\frac{s}{2}}$ where $s \in (0,1)$. There we first use the integral representation of $\Delta^{\frac{t}{2}}$ as in Section 2.3 and then apply in turns the fundamental theorem of calculus and the mean value condition.

3.1 On the Ball

Proposition 3.3. *Let $\gamma \in [0, n+2)$, $N \in \mathbb{N}$. Then for a constant $C_{N,\gamma}$ and for any $v \in C^\infty(\mathbb{R}^n)$ satisfying (3.1) on some $D = B_r \subset \mathbb{R}^n$,*

$$\int_{B_r} \int_{B_r} \frac{|v(x)-v(y)|^2}{|x-y|^\gamma} \, dy \, dx \le C_{N,\gamma} \ r^{2N-\gamma} \int_{B_r} \int_{B_r} \left|\nabla^N v(x) - \nabla^N v(y)\right|^2 \, dx \, dy.$$

***Proof of Proposition* 3.3.**
It suffices to prove this proposition for $B_1(0)$ and then scale the estimate. So let $r = 1$. By Proposition 3.1,

$$\begin{aligned}\int_{B_1}\int_{B_1} \frac{|v(x)-v(y)|^2}{|x-y|^{\gamma}}\,dy\,dx &\prec \int_{B_1} |\nabla v(z)|^2\,dz \\ &\overset{(3.1)}{=} \int_{B_1} |\nabla v(z) - (\nabla v)_{B_1}|^2\,dz \\ &\prec \int_{B_1}\int_{B_1} |\nabla v(z) - \nabla v(z_2)|^2\,dz\,dz_2\end{aligned}$$

Iterating this procedure N times with repeated use of Proposition 3.1 for $\gamma = 0$, we conclude.

Proposition 3.3 □

Proposition 3.4. *For any $N \in \mathbb{N}_0$, $s \in [0,1)$ there is a constant $C_{N,s} > 0$ such that the following holds. For any $v \in C^\infty(\mathbb{R}^n)$, $r > 0$, $x_0 \in \mathbb{R}^n$ such that (3.1) holds on $D = B_{4r}(x_0)$ we have for all multiindices α, $\beta \in (\mathbb{N}_0)^n$, $|\alpha| + |\beta| = N$*

$$\left\|\Delta^{\frac{s}{2}}\left((\partial^\alpha \eta_{r,x_0})(\partial^\beta v)\right)\right\|_{L^2(\mathbb{R}^n)} \le C_{N,s}\,[v]_{B_{4r}(x_0),N+s}.$$

***Proof of Proposition* 3.4.**
The case $s = 0$ follows by the classic Poincaré inequality, so let from now on $s \in (0,1)$. Set

$$w(y) := (\partial^\alpha \eta_r(y))(\partial^\beta v(y)).$$

Note that $\operatorname{supp} w \subset B_{2r}$. Moreover, by the definition of η_r, we have

$$|w| \le C_\alpha\, r^{-|\alpha|}\left|\partial^\beta v\right| \le C_N r^{|\beta|-N}\left|\partial^\beta v\right|. \tag{3.2}$$

By Proposition 2.10 we have to estimate

$$\begin{aligned}\|\Delta^{\frac{s}{2}} w\|_{L^2}^2 &\approx \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ &= \int_{B_{4r}}\int_{B_{4r}} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + 2\int_{B_{4r}} |w(y)|^2 \int_{\mathbb{R}^n\setminus B_{4r}} \frac{1}{|x-y|^{n+2s}}\,dx\,dy =: I + 2II.\end{aligned}$$

To estimate II, we use the fact that $\operatorname{supp} w \subset B_{2r}$ to get

$$\begin{aligned}|II| &\prec r^{-2s}\int_{B_{4r}} |w(y)|^2\,dy \overset{(3.2)}{\prec} r^{2(|\beta|-N-s)}\int_{B_{4r}} \left|\partial^\beta v(y)\right|^2\,dy \\ &\overset{(3.1)}{\prec} r^{2(|\beta|-N-s)-n}\int_{B_{4r}}\int_{B_{4r}} \left|\partial^\beta v(y) - \partial^\beta v(x)\right|^2\,dy\,dx.\end{aligned}$$

As $\partial^\beta v$ satisfies (3.1) for $N - |\beta|$, by Proposition 3.3 for $\gamma = 0$,

$$\int_{B_{4r}}\int_{B_{4r}} \left|\partial^\beta v(y) - \partial^\beta v(x)\right|^2\,dy\,dx \prec r^{2(N-|\beta|)}\int_{B_{4r}}\int_{B_{4r}} \left|\nabla^N v(y) - \nabla^N v(x)\right|^2\,dx\,dy.$$

Furthermore, for $x, y \in B_{4r}$ we have $r^{-n-2s} \prec |x-y|^{-n-2s}$ which altogether implies that

$$|II| \prec [v]_{B_{4r},N+s}.$$

In order to estimate I, note that

$$|w(x) - w(y)| \prec r^{-|\alpha|}\left|\partial^\beta v(x) - \partial^\beta v(y)\right| + r^{-|\alpha|-1}|x-y|\,\left|\partial^\beta v(y)\right|.$$

Thus, we can decompose $|I| \prec |I_1| + |I_2|$ where

$$I_1 = r^{2(|\beta|-N)} \int\limits_{B_{4r}} \int\limits_{B_{4r}} \frac{\left|\partial^\beta v(x) - \partial^\beta v(y)\right|^2}{|x-y|^{n+2s}} \, dx \, dy,$$

and

$$\begin{aligned} I_2 &= r^{2(|\beta|-N-1)} \int\limits_{B_{4r}} \int\limits_{B_{4r}} \frac{\left|\partial^\beta v(y)\right|^2}{|x-y|^{n-2+2s}} \, dx \, dy \overset{s<1}{\prec} r^{2(|\beta|-N)-2s} \int\limits_{B_{4r}} \left|\partial^\beta v(y)\right|^2 dy \\ &\overset{(3.1)}{\prec} r^{2(|\beta|-N)-(n+2s)} \int\limits_{B_{4r}} \int\limits_{B_{4r}} \left|\partial^\beta v(y) - \partial^\beta v(z)\right|^2 dy \, dz. \end{aligned}$$

Using again that $\partial^\beta v$ satisfies (3.1) for $N - |\beta|$ on B_{4r}, by Proposition 3.3 for $\gamma = n + 2s$

$$|I_1| \prec r^{-n-2s} \int\limits_{B_{4r}} \int\limits_{B_{4r}} \left|\nabla^N u(x) - \nabla^N u(y)\right|^2 dx \, dy \prec \int\limits_{B_{4r}} \int\limits_{B_{4r}} \frac{\left|\nabla^N u(x) - \nabla^N u(y)\right|^2}{|x-y|^{n+2s}} \, dx \, dy,$$

and the same for I_2. This concludes the case $s > 0$.

Proposition 3.4 □

Lemma 3.5 (Poincaré inequality with mean value condition (Ball))**.** *For any $N \in \mathbb{N}_0$, $s \in [0, N+1)$, $t \in [0, N+1-s)$ there is a constant $C_{N,s,t}$ such that the following holds. For any $r > 0$, $x_0 \in \mathbb{R}^n$ and any $v \in C^\infty(\mathbb{R}^n)$ satisfying (3.1) for N and on $D = B_{4r}(x_0)$, we have*

$$\|\Delta^{\frac{s}{2}} \eta_{r,x_0} v\|_{L^2(\mathbb{R}^n)} \leq C_{s,t} \; r^t \; [v]_{B_{4r}(x_0),s+t} \leq C_{s,t} \; r^t \|\Delta^{\frac{s+t}{2}} v\|_{L^2(\mathbb{R}^n)}.$$

Proof of Lemma* 3.5*.
We have $\Delta^{\frac{s}{2}} \approx \Delta^{\frac{\gamma}{2}} \Delta^{\frac{\delta}{2}} \Delta^K$ for $\gamma = s - \lfloor s \rfloor \in [0,1)$, $\delta = \lfloor s \rfloor - 2 \left\lfloor \frac{\lfloor s \rfloor}{2} \right\rfloor \in \{0,1\}$, and $K = \left\lfloor \frac{\lfloor s \rfloor}{2} \right\rfloor \in \mathbb{N}_0$. As the Riesz Transform $\mathcal{R}_i$ is a bounded operator from L^2 into L^2 we can estimate

$$\|\Delta^{\frac{s}{2}} (\eta_r v)\|_{L^2} \prec \sum_{\substack{\alpha,\beta \in (\mathbb{N}_0)^n \\ |\alpha|+|\beta| = 2K+\delta}} \|\Delta^{\frac{\gamma}{2}} \left((\partial^\alpha \eta_r)(\partial^\beta v)\right) \|_{L^2} \overset{P.3.4}{\prec} \left([v]_{B_{4r}(x_0),s}\right)^2.$$

If $t = 0$ this gives the claim. So let now $t > 0$. For every $s > 0$ we have (using possibly the mean value condition if $s \in \mathbb{N}$)

$$[v]^2_{B_{4r}(x_0),s} \prec \int\limits_{B_{4r}} \int\limits_{B_{4r}} \frac{\left(\nabla^{\lfloor s \rfloor} u(x) - \nabla^{\lfloor s \rfloor} u(y)\right)^2}{|x-y|^{n+2(s-\lfloor s \rfloor)}} \, dx \, dy.$$

If $\lfloor s \rfloor = \lfloor s + t \rfloor$, this implies using $|x - y| \succ r$ for $x, y \in B_{4r}$,

$$[v]^2_{B_{4r}(x_0),s} \prec r^{2t} [v]^2_{B_{4r}(x_0),s+t}.$$

Possibly using Proposition 3.3 one concludes.

Lemma 3.5 □

By obvious modifications of the proofs, one checks that the result of Lemma 3.5 is also valid if v satisfies (3.1) on a ball $B_{\lambda r}$ for $\lambda \in (0, 4)$. The constant then depends also on λ.

3.2 On the Annulus

By similar methods, covering an Annulus by Family of convex sets without enlarging it too much, we can prove the following lemma. For a proof we refer to [Sch10b].

Lemma 3.6 (Poincaré's Inequality with mean value condition (Annulus)). *For any $N \in \mathbb{N}_0$, $s \in [0, N+1)$, $t \in [0, N+1-s)$ there is a constant $C_{N,s,t}$ such that the following holds. For any $v \in C^\infty(\mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$, $r > 0$ such that v satisfies (3.1) for N on $D = A_k = B_{2^{k+1}r}(x_0)\backslash B_{2^{k-1}r}(x_0)$ or $D = A_k = B_{2^{k+1}r}(x_0)\backslash B_{2^k r}(x_0)$ we have*

$$\|\Delta^{\frac{s}{2}}(\eta^k_{r,x_0} v)\|_{L^2(\mathbb{R}^n)} \le C_{s,t}\ \left(2^k r\right)^t\ [v]_{\tilde{A}_k, s+t},$$

where

$$\tilde{A}_k = B_{2^{k+2}r}(x_0)\backslash B_{2^{k-2}r}(x_0).$$

Again, one checks that the claim is also satisfied if v satisfies (3.1) on a possibly smaller annulus, making the constant depending also on this scaling.

3.3 Comparison between Mean Value Polynomials on Different Sets

For a bounded domain $D \subset \mathbb{R}^n$ and $N \in \mathbb{N}_0$ and for $v \in \mathcal{S}(\mathbb{R}^n)$ we define the polynomial $P(v) \equiv P_{D,N}(v)$ to be the unique polynomial of order N such that

$$\fint_D \partial^\alpha (v - P(v)) = 0, \quad \text{for every multiindex } \alpha \in (\mathbb{N}_0)^n,\ |\alpha| \le N. \tag{3.3}$$

The goal of this section is to estimate in Proposition 3.10 and Lemma 3.12 the difference

$$P_{B_r(x),N}(v) - P_{B_{2^k r}(x)\backslash B_{2^{k-1}r}(x),N}(v), \quad \text{for } k \in \mathbb{Z}$$

in terms of $\Delta^{\frac{s}{2}} v$. To do so, we adapt the methods applied in the proof of [DLR09, Lemma 4.2], the main difference being that we have to extend their argument to polynomials of degree greater than zero. We will need an inductive description of $P(v)$. As stated in the introduction, for a multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$ we set $\alpha! := \alpha_1! \dots \alpha_n! = \partial^\alpha x^\alpha$. For $i \in \{0, \dots, N\}$ we denote

$$\begin{aligned} Q^i_{D,N}(v) &:= Q^{i+1}_{D,N}(v) + \sum_{|\alpha|=i} \frac{1}{\alpha!}\ x^\alpha \fint_D \partial^\alpha (v - Q^{i+1}_{D,N}(v)),\\ Q^N_{D,N}(v) &:= \sum_{|\alpha|=N} \frac{1}{\alpha!}\ x^\alpha \fint_D \partial^\alpha v. \end{aligned} \tag{3.4}$$

One checks that

$$\partial^\alpha Q^i = \partial^\alpha P, \quad \text{whenever } |\alpha| \ge i, \tag{3.5}$$

and in particular $Q^0 = P$.
Moreover we will introduce the following sets of annuli (Note that in other sections A_j, $\tilde{A}_j$ might denote different annuli):

$$A_j \equiv A_j(r) = B_{2^j r}\backslash B_{2^{j-1}r}, \quad \tilde{A}_j \equiv \tilde{A}_j(r) := A_j \cup A_{j+1}.$$

Proposition 3.7. *For any $N \in \mathbb{N}$, $s \in (N, N+1]$, $D \subseteq D_2 \subset \mathbb{R}^n$ smoothly bounded domains there is a constant $C_{D_2,D,N,s}$ such that the following holds: Let $v \in C^\infty(\mathbb{R}^n)$. For any multiindex $\alpha \in (\mathbb{N}_0)^n$ such that $|\alpha| = i \le N-1$,*

$$\int_{D_2} \left|\partial^\alpha (v - Q^{i+1}_{D,N}(v)) - \left(\partial^\alpha (v - Q^{i+1}_{D,N}(v))\right)_D\right| \le C_{D_2,D,N,s}\ \left(\frac{|D_2|}{|D|}\right)^{\frac{1}{2}} \operatorname{diam}(D_2)^{\frac{n}{2}+s-N}\ [v]_{D_2,s}$$

where $[v]_{D,s}$ is defined as in (2.3). If $D = r\tilde{D}$, $D_2 = r\tilde{D}_2$, then $C_{D_2,D,N,s} = r^{N-i} C_{\tilde{D}_2,\tilde{D},N,s}$.

Proof of Proposition 3.7.
Let us denote

$$I := \int_{D_2} \left|\partial^\alpha (v - Q^{i+1}_{D,N}) - \left(\partial^\alpha (v - Q^{i+1}_{D,N}(v))\right)_D\right|.$$

A first application of Hölder's and classic Poincaré's inequality yields

$$I \le C_{D,D_2}\ |D_2|^{\frac{1}{2}}\ \|\nabla \partial^\alpha (v - Q^{i+1}_{D,N})\|_{L^2(D_2)}.$$

Next, (3.5) and the definition of P in (3.3) imply that we can apply classic Poincaré inequality $N-i-1$ times more, to estimate I by

$$\le C_{D_2,D,N}\ |D_2|^{\frac{1}{2}}\ \|\nabla^N(v-P_{D,N}(v))\|_{L^2(D_2)} \overset{(3.4)}{=} C_{D_2,D,N}\ |D_2|^{\frac{1}{2}}\ \|\nabla^N v - \left(\nabla^N v\right)_D\|_{L^2(D_2)}.$$

If $s=N+1$, yet another application of Poincaré's inequality yields the claim. In the case $s\in(N,N+1)$, we estimate further

$$I \quad \le \quad C_{D_2,D,N}\ \left(\frac{|D_2|}{|D|}\right)^{\frac{1}{2}}\ \left(\int_{D_2}\int_{D_2} \left|\nabla^N v(x)-\nabla^N v(y)\right|^2\,dx\,dy\right)^{\frac{1}{2}},$$

which is bounded by

$$C_{D_2,D,N}\ \left(\frac{|D_2|}{|D|}\right)^{\frac{1}{2}}\ \operatorname{diam}(D_2)^{\frac{n+2(s-N)}{2}}\ \left(\int_{D_2}\int_{D_2} \frac{\left|\nabla^N v(x)-\nabla^N v(y)\right|^2}{|x-y|^{n+2(s-N)}}\,dx\,dy\right)^{\frac{1}{2}}.$$

The scaling factor for $D=r\tilde{D}$ then follows by the according scaling factors of Poincaré's inequality.

Proposition 3.7 □

Proposition 3.8. *For any $N\in\mathbb{N}_0$, $s\in(N,N+1]$, there is a constant $C_{N,s}>0$ such that the following holds: For any $j\in\mathbb{Z}$, any multiindex $|\alpha|\le i\le N$ and $v\in C^\infty(\mathbb{R}^n)$*

$$\left\|\partial^\alpha\left(Q^i_{A_j,N}-Q^i_{A_{j+1},N}\right)\right\|_{L^\infty(A_j)} \le C_{N,s}(2^j r)^{s-|\alpha|-\frac{n}{2}}\ [v]_{\tilde{A}_j,s}.$$

***Proof of Proposition* 3.8.**
Assume first that $i=N$. Then if $s\in(N,N+1)$,

$$\|\partial^\alpha(Q^N_{A_j}-Q^N_{A_{j+1}})\|_{L^\infty(A_j)} \overset{(3.4)}{\prec} (2^j r)^{N-|\alpha|}\frac{1}{|A_j|^2}\int_{\tilde{A}_j}\int_{\tilde{A}_j}\left|\nabla^N v(x)-\nabla^N v(y)\right|\,dx\,dy \prec (2^j r)^{-|\alpha|-\frac{n}{2}+s}[v]_{\tilde{A}_j,s}.$$

If $s=N+1$ and $i=N$, one uses classic Poincaré inequality to prove the claim.
Now let $i\le N-1$, $s\in(N,N+1]$, and assume we have proven the claim for $i+1$. By (3.4),

$$\begin{aligned} Q^i_{A_j}-Q^i_{A_{j+1}} \quad &= \quad Q^{i+1}_{A_j}-Q^{i+1}_{A_{j+1}} \\ &\quad + \sum_{|\beta|=i}\frac{1}{\beta!}\,x^\beta\left(\fint_{A_j}\partial^\beta(v-Q^{i+1}_{A_{j+1}}) - \fint_{A_{j+1}}\partial^\beta(v-Q^{i+1}_{A_{j+1}})\right) \\ &\quad + \sum_{|\beta|=i}\frac{1}{\beta!}\,x^\beta\left(\fint_{A_j}\partial^\beta(Q^{i+1}_{A_{j+1}}-Q^{i+1}_{A_j})\right). \end{aligned}$$

Consequently,

$$\begin{aligned} \|\partial^\alpha(Q^i_{A_j}-Q^i_{A_{j+1}})\|_{L^\infty(A_j)} \quad &\prec \quad \|\partial^\alpha(Q^{i+1}_{A_j}-Q^{i+1}_{A_{j+1}})\|_{L^\infty(A_j)} \\ &\quad +(2^j r)^{i-|\alpha|}\sum_{|\beta|=i}\fint_{A_j}\left|\partial^\beta(v-Q^{i+1}_{A_{j+1}}) - \fint_{A_{j+1}}\partial^\beta(v-Q^{i+1}_{A_{j+1}})\right| \\ &\quad +(2^j r)^{i-|\alpha|}\sum_{|\beta|=i}\|\partial^\beta(Q^{i+1}_{A_{j+1}}-Q^{i+1}_{A_j})\|_{L^\infty(A_j)}. \end{aligned}$$

Then the claim for $i+1$ and Proposition 3.7 conclude the proof.

Proposition 3.8 □

Proposition 3.9. *For any $N \in \mathbb{N}_0$, $s \in (N, N+1]$ there is a constant $C_{N,s}$ such that the following holds. For any multiindex $\alpha \in (\mathbb{N}_0)^n$, $|\alpha| \leq i \leq N$, for any $r > 0$, $k \in \mathbb{Z}$ and any $v \in \mathcal{S}(\mathbb{R}^n)$ if $s - \frac{n}{2} \notin \{i, \ldots, N\}$,*

$$\|\partial^\alpha (Q^i_{B_r} - Q^i_{A_k})\|_{L^\infty(\tilde{A}_k)} \leq C_{N,s}\ r^{s-|\alpha|-\frac{n}{2}} \left(2^{k(s-|\alpha|-\frac{n}{2})} + 2^{k(i-|\alpha|)}\right)\ [v]_{\mathbb{R}^n,s},$$

and if $s - \frac{n}{2} \in \{i, \ldots, N\}$,

$$\|\partial^\alpha (Q^i_{B_r} - Q^i_{A_k})\|_{L^\infty(\tilde{A}_k)} \leq C_{N,s}\ r^{s-|\alpha|-\frac{n}{2}}\ 2^{k(i-|\alpha|)} \left(|k| + 1 + 2^{k(s-i-\frac{n}{2})}\right)\ [v]_{\mathbb{R}^n,s}.$$

Here as before, $A_k = B_{2^k r}(x) \backslash B_{2^{k-1}r}(x)$ and $\tilde{A}_k = B_{2^{k+1}r}(x) \backslash B_{2^{k-1}r}(x)$.

Proof of Proposition 3.9.
For the sake of shortness of presentation, let us abbreviate

$$d^{i,\alpha}_k := \|\partial^\alpha (Q^i_{B_r} - Q^i_{A_k})\|_{L^\infty(\tilde{A}_k)}.$$

Assume first $i = N$.

$$\begin{aligned}
d^{N,\alpha}_k &\overset{(3.4)}{\prec} \left\| \sum_{|\beta|=N} \frac{\partial^\alpha x^\beta}{\beta!} \left(\fint_{B_r} \partial^\beta v - \fint_{A_k} \partial^\beta v \right) \right\|_{L^\infty(\tilde{A}_k)} \\
&\prec (2^k r)^{N-|\alpha|} \left| \fint_{B_r} \nabla^N v - \fint_{A_k} \nabla^N v \right| \approx (2^k r)^{N-|\alpha|} \left| \sum_{l=-\infty}^{0} \frac{|A_l|}{|B_r|} \fint_{A_l} \nabla^N v - \fint_{A_k} \nabla^N v \right|.
\end{aligned}$$

As $\frac{|A_l|}{|B_r|} = 2^{ln}(1 - 2^{-n})$ and thus $\sum\limits_{l=-\infty}^{0} \frac{|A_l|}{|B_r|} = 1$, for $k > 0$ we estimate further

$$\begin{aligned}
d^{N,\alpha}_k &\prec (2^k r)^{N-|\alpha|} \sum_{l=-\infty}^{0} 2^{ln} \left| \fint_{A_l} \nabla^N v - \fint_{A_k} \nabla^N v \right| \prec (2^k r)^{N-|\alpha|} \sum_{l=-\infty}^{0} 2^{ln} \sum_{j=l}^{k-1} \left| \fint_{A_j} \nabla^N v - \fint_{A_{j+1}} \nabla^N v \right| \\
&\overset{(\star)}{\prec} (2^k r)^{N-|\alpha|} \sum_{l=-\infty}^{0} 2^{ln} \sum_{j=l}^{k-1} (2^j r)^{-n} \left(\int_{\tilde{A}_j} \int_{\tilde{A}_j} \left|\nabla^N v(x) - \nabla^N v(y)\right|^2\, dx\, dy \right)^{\frac{1}{2}} \\
&\prec (2^k r)^{N-|\alpha|} \sum_{l=-\infty}^{0} 2^{ln} \sum_{j=l}^{k-1} (2^j r)^{-\frac{n}{2}+s-N}\ [v]_{\tilde{A}_j,s}.
\end{aligned}$$

Of course, if $s = N + 1$, one replaces the estimate in ($\bigstar$) and uses instead Poincaré's inequality. If $k \leq 0$ one has by virtually the same computation,

$$d^{N,\alpha}_k \prec (2^k)^{N-|\alpha|} r^{s-\frac{n}{2}-|\alpha|} \left(\sum_{l=-\infty}^{k-1} 2^{ln} \sum_{j=l}^{k-1} 2^{j(-\frac{n}{2}+s-N)}\ [v]_{\tilde{A}_j,s} + \sum_{l=k}^{0} 2^{ln} \sum_{j=k}^{l-1} 2^{j(-\frac{n}{2}+s-N)}\ [v]_{\tilde{A}_j,s} \right).$$

Now we have to take care, whether $s - \frac{n}{2} - N = 0$ or not. Let

$$a_k := \begin{cases} 2^{k(s-\frac{n}{2}-N)}, & \text{if } s - \frac{n}{2} - N \neq 0, \\ |k|, & \text{if } s - \frac{n}{2} - N = 0, \end{cases}, \quad b_l := \begin{cases} 2^{l(s-\frac{n}{2}-N)}, & \text{if } s - \frac{n}{2} - N \neq 0, \\ |l|, & \text{if } s - \frac{n}{2} - N = 0. \end{cases}$$

Then, applying Hölder's inequality for series, $d^{N,\alpha}_k$ is estimated independently of whether $k > 0$ or not, by

$$\begin{aligned}
&(2^k)^{N-|\alpha|} r^{s-|\alpha|-\frac{n}{2}} \sum_{l=-\infty}^{0} 2^{ln}\, (a_k + b_l) \left(\sum_{j=-\infty}^{\infty} [v]^2_{\tilde{A}_j,s} \right)^{\frac{1}{2}} \\
\prec\ & r^{s-\frac{n}{2}-|\alpha|} \left(2^{k(N-|\alpha|)} a_k + (2^k)^{N-|\alpha|} \sum_{l=-\infty}^{0} 2^{ln} b_l \right) [v]_{\mathbb{R}^n,s} \\
\prec\ & r^{s-\frac{n}{2}-|\alpha|}\ [v]_{\mathbb{R}^n,s} \left(2^{k(N-|\alpha|)} a_k + 2^{k(N-|\alpha|)} \right).
\end{aligned}$$

This concludes the case $i = N$. Next, let $i < N$ and assume the claim is proven for $i+1$.

$$\begin{aligned} d_k^{i,\alpha} &= \|\partial^\alpha (Q^i_{B_r} - Q^i_{A_k})\|_{L^\infty(\tilde{A}_k)} \\ &\overset{(3.4)}{\prec} d_k^{i+1,\alpha} + \sum_{|\beta|=i} \left(2^k r\right)^{i-|\alpha|} \left| \fint_{B_r} \partial^\beta (v - Q^{i+1}_{B_r}) - \fint_{A_k} \partial^\beta (v - Q^{i+1}_{A_k}) \right| \\ &\prec d_k^{i+1,\alpha} + \sum_{|\beta|=i} \left(2^k r\right)^{i-|\alpha|} c_n \sum_{l=-\infty}^{0} 2^{ln} \left| \fint_{A_l} \partial^\beta (v - Q^{i+1}_{B_r}) - \fint_{A_k} \partial^\beta (v - Q^{i+1}_{A_k}) \right|, \end{aligned}$$

where $c_n 2^{ln} = \frac{|A_l|}{|B_r|}$, so $\sum\limits_{l=-\infty}^{0} c_n 2^{ln} = 1$ as we have done in the case $i = N$ above. We estimate further,

$$d_k^{i,\alpha} \prec d_k^{i+1,\alpha} + \sum_{|\beta|=i} \left(2^k r\right)^{i-|\alpha|} \sum_{l=-\infty}^{0} 2^{ln} \left(d_l^{i+1,\beta} + \left| \fint_{A_l} \partial^\beta (v - Q^{i+1}_{A_l}) - \fint_{A_k} \partial^\beta (v - Q^{i+1}_{A_k}) \right| \right).$$

As above in the case $i = N$ we use a telescoping series to write

$$\begin{aligned} &\left| \fint_{A_l} \partial^\beta (v - Q^{i+1}_{A_l}) - \fint_{A_k} \partial^\beta (v - Q^{i+1}_{A_k}) \right| \\ \prec &\sum_{j=l}^{k-1} \left(\left\| \partial^\beta (Q^{i+1}_{A_j} - Q^{i+1}_{A_{j+1}}) \right\|_{L^\infty(A_j)} + \fint_{\tilde{A}_j} \left| \partial^\beta (v - Q^{i+1}_{A_{j+1}}) - \fint_{A_{j+1}} \partial^\beta (v - Q^{i+1}_{A_{j+1}}) \right| \right) =: \sum_{j=l}^{k-1} (I_j + II_j). \end{aligned}$$

Again we should have taken care of whether $l < k-1$ or $k-1 \leq l$, but as in the case $i = N$ both cases are treated the same way. The term I_j, II_j are estimated by Proposition 3.8 and Proposition 3.7,

$$I_j \prec \left(2^j r\right)^{s-|\beta|-\frac{n}{2}} \; [v]_{\tilde{A}_j,s} = \left(2^j r\right)^{s-i-\frac{n}{2}} \; [v]_{\tilde{A}_j,s}.$$

$$II_j \prec (2^j r)^{-n+\frac{n}{2}+s-i} \; [v]_{\tilde{A}_j,s} = (2^j r)^{s-i-\frac{n}{2}} \; [v]_{\tilde{A}_j,s}.$$

Hence,

$$\left| \fint_{A_l} \partial^\beta (v - Q^{i+1}_{A_l}) - \fint_{A_k} \partial^\beta (v - Q^{i+1}_{A_k}) \right| \prec r^{s-i-\frac{n}{2}} \sum_{j=l}^{k-1} (2^j)^{s-i-\frac{n}{2}} \; [v]_{\tilde{A}_j,s} \prec r^{s-i-\frac{n}{2}} \, (a_k + b_l) \left(\sum_{j=l}^{k-1} [v]^2_{\tilde{A}_j,s} \right)^{\frac{1}{2}},$$

for a_k and b_k similar to the case $i = N$ above defined as

$$a_k := \begin{cases} 2^{k(s-\frac{n}{2}-i)}, & \text{if } s - \frac{n}{2} - i \neq 0, \\ |k|, & \text{if } s - \frac{n}{2} - i = 0, \end{cases} \quad b_l := \begin{cases} 2^{l(s-\frac{n}{2}-i)}, & \text{if } s - \frac{n}{2} - i \neq 0, \\ |l|, & \text{if } s - \frac{n}{2} - i = 0. \end{cases}$$

Plugging all these estimates in, we have achieved the following estimate

$$d_k^{i,\alpha} \prec d_k^{i+1,\alpha} + \sum_{|\beta|=i} \left(2^k r\right)^{i-|\alpha|} \sum_{l=-\infty}^{0} 2^{ln} d_l^{i+1,\beta} + r^{s-|\alpha|-\frac{n}{2}} \; 2^{k(i-|\alpha|)} \; (a_k + 1) \; [v]_{\mathbb{R}^n,s}.$$

In either case, whether $s - \frac{n}{2} - \tilde{i} = 0$ for some $\tilde{i} \geq i$ or not, using the claim for $i+1$ we have

$$\sum_{|\beta|=i} \left(2^k r\right)^{i-|\alpha|} \sum_{l=-\infty}^{0} 2^{ln} d_l^{i+1,\beta} \prec C_{N,s} \; r^{s-\frac{n}{2}-|\alpha|} [v]_{\mathbb{R}^n,s},$$

Proposition 3.9 □

As an immediate consequence of Proposition 3.9 for $i = 0$, $|\alpha| = 0$, and $s = \frac{n}{2}$, we get the following two results.

Proposition 3.10. *For a uniform constant $C > 0$, for any $v \in \mathcal{S}(\mathbb{R}^n)$, $r > 0$, $k \in \mathbb{N}$*

$$\|\eta_r^k (P_{B_r,\lceil \frac{n}{2}\rceil -1}(v) - P_{A_k,\lceil \frac{n}{2}\rceil -1}(v))\|_{L^\infty(\mathbb{R}^n)} \leq C\ (1+|k|)\|\Delta^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)}.$$

Here, $A_k = B_{2^{k+1}r}(x)\backslash B_{2^k r}(x)$ and $\tilde{A}_k = B_{2^{k+1}r}(x)\backslash B_{2^{k-1}r}(x)$.

Proposition 3.11. *There exists a constant $C > 0$ such that for any $r > 0$, $x_0 \in \mathbb{R}^n$, $k \in \mathbb{N}_0$, $v \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$\|\eta_{r,x_0}^k (v - P)\|_{L^2(\mathbb{R}^n)} \leq C \left(2^k r\right)^{\frac{n}{2}}\ (1+|k|)\ \|\Delta^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)},$$

where P is the polynomial of order $N := \lceil \frac{n}{2}\rceil - 1$ such that $v - P$ satisfies the mean value condition (3.1) in $D := B_{2r}$. Here, in a slight abuse of notation for $k = 0$, $\eta_r^k \equiv \eta_r - \eta_{\frac{1}{2}r}$ for η from Section 2.4.

***Proof of Proposition* 3.11.**
Let P_k be the polynomial of order $N = \lceil \frac{n}{2}\rceil - 1$ such that v satisfies the mean value condition (3.1) in $B_{2^k r}\backslash B_{2^{k-1}r}$. We then have,

$$\|\eta_r^k (v - P)\|_{L^2(\mathbb{R}^n)} \prec \|\eta_r^k (v - P_k)\|_{L^2(\mathbb{R}^n)} + \left(2^k r\right)^{\frac{n}{2}} \|\eta_r^k (P - P_k)\|_{L^\infty}.$$

As Proposition 3.10 estimates the second part of the last estimate, we are left to estimate

$$\|\eta_r^k (v - P_k)\|_{L^2(\mathbb{R}^n)} \leq C \left(2^k r\right)^{\frac{n}{2}}\ \|\Delta^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)}.$$

But this is rather easy and can be proven by similar arguments as used in the proof of Lemma 3.6: as by classic Poincaré inequality and the fact that by choice of P_k the mean values over $B_{2^{k+1}r}\backslash B_{2^k r}$ of all derivatives up to order $\lfloor \frac{n}{2}\rfloor$ of $v - P_k$ are zero, so

$$\|\eta_r^k (v - P_k)\|_{L^2(\mathbb{R}^n)} \prec \left(2^k r\right)^{\lfloor \frac{n}{2}\rfloor}\ \|\nabla^{\lfloor \frac{n}{2}\rfloor}(v - P_k)\|_{L^2(B_{2^{k+1}r}\backslash B_{2^{k-1}r})}.$$

If n is an even number, this proves the claim. If n is odd, we use again the mean value condition to see

$$\begin{aligned}
\|\nabla^N (v - P_k)\|^2_{L^2(B_{2^{k+1}r}\backslash B_{2^{k-1}r})} &\prec \fint_{B_{2^{k+1}r}\backslash B_{2^k r}} \int_{B_{2^{k+1}r}\backslash B_{2^{k-1}r}} \left|\nabla^N v(x) - \nabla^N v(y)\right|^2\, dx\, dy \\
&\prec \left(2^k r\right)^{n - 2\lfloor \frac{n}{2}\rfloor}\ \|\Delta^{\frac{n}{4}} v\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Taking the square root of the last estimate, one concludes.

Proposition 3.11 □

We will need the following a little bit sharper version of Proposition 3.10, too. The interested reader might compare what follows to [DLR09, Lemma 4.2] which is a special case of the next result.

Lemma 3.12. *Let $N := \lceil \frac{n}{2}\rceil - 1$ and $\gamma > N$. Then for $\tilde{\gamma} = -N + \min(n, \gamma)$ and for any $v \in \mathcal{S}(\mathbb{R}^n)$, $B_r(x_0) \subset \mathbb{R}^n$, $r > 0$,*

$$\sum_{k=1}^{\infty} 2^{-\gamma k}\, \|(P_{B_r,N}(v) - P_{A_k,N}(v))\|_{L^\infty(\tilde{A}_k)} \leq C_\gamma \sum_{j=-\infty}^{\infty} 2^{-|j|\tilde{\gamma}}\ [v]_{\tilde{A}_j,\frac{n}{2}}.$$

Here, $A_k = B_{2^{k+1}r}(x)\backslash B_{2^k r}(x)$ and $\tilde{A}_k = B_{2^{k+1}r}(x)\backslash B_{2^{k-1}r}(x)$.

More precisely, we will prove for $i \in \{0, \ldots, N\}$, that whenever $\gamma > N$, $|\alpha| \leq i$, for $\tilde{\gamma} := \min(n - N, \gamma - N)$

$$\sum_{k=-\infty}^{\infty} 2^{-\gamma|k|} \|\partial^\alpha (Q^i_{B_r} - Q^i_{A_k})\|_{L^\infty(\tilde{A}_k)} \leq C_{\gamma,N} \left(r^{-|\alpha|} \sum_{j=-\infty}^{\infty} 2^{-|j|\tilde{\gamma}}\ [v]_{\tilde{A}_j,\frac{n}{2}} \right). \tag{3.6}$$

This more precise statement will be used in the estimates for the homogeneous norm $[\cdot]_s$, Lemma 8.1.
***Proof of Lemma* 3.12.**
As in the proof of Proposition 3.9, set $d_k^{i,\alpha} := \|\partial^\alpha (Q^i_{B_r} - Q^i_{A_k})\|_{L^\infty(\tilde{A}_k)}$. Moreover, we set

$$S_\gamma^{i,\alpha} := \sum_{k=1}^{\infty} 2^{-\gamma k}\ d_k^{i,\alpha} \quad \text{and} \quad S_{-\gamma}^{i,\alpha} := \sum_{k=-\infty}^{0} 2^{\gamma k}\ d_k^{i,\alpha}.$$

We will only treat the case $S^{i,\alpha}_{\gamma}$, the case of $S^{i,\alpha}_{\gamma}$ is done analogously. By the computations in the proof of Proposition 3.9, for any $|\alpha| \leq N$,

$$\begin{aligned}
& S^{N,\alpha}_{\gamma} \\
\prec\ & r^{-|\alpha|} \sum_{k=1}^{\infty} \sum_{l=-\infty}^{0} \sum_{j=l}^{k-1} 2^{-jN+ln-\gamma k+kN-k|\alpha|}\, [v]_{\tilde{A}_j,\frac{n}{2}} \\
=\ & r^{-|\alpha|} \sum_{j=-\infty}^{0} 2^{-jN}\, [v]_{\tilde{A}_j,\frac{n}{2}} \sum_{l=-\infty}^{j} \sum_{k=1}^{\infty} 2^{ln}\, 2^{k(N-\gamma-|\alpha|)} \\
& + r^{-|\alpha|} \sum_{j=1}^{\infty} 2^{-jN}\, [v]_{\tilde{A}_j,\frac{n}{2}} \sum_{l=-\infty}^{0} \sum_{k=j+1}^{\infty} 2^{ln}\, 2^{k(N-\gamma-|\alpha|)} \\
\overset{\gamma>N}{\prec}\ & r^{-|\alpha|} \sum_{j=-\infty}^{0} 2^{j(n-N)}\, [v]_{\tilde{A}_j,\frac{n}{2}} + r^{-|\alpha|} \sum_{j=1}^{\infty} 2^{j(-\gamma-|\alpha|)}\, [v]_{\tilde{A}_j,\frac{n}{2}}.
\end{aligned}$$

For $0 \leq i \leq N-1$, using the computations done for the proof of Proposition 3.9,

$$\begin{aligned}
S^{i,\alpha}_{\gamma} \prec\ & S^{i+1,\alpha}_{\gamma} + r^{i-|\alpha|} \sum_{|\beta|=i} \sum_{k=1}^{\infty} 2^{k(i-|\alpha|-\gamma)} S^{i+1,\beta}_{-n} + r^{-|\alpha|} \sum_{k=1}^{\infty} 2^{k(i-|\alpha|-\gamma)} \sum_{l=-\infty}^{0} 2^{ln} \sum_{j=l}^{k-1} 2^{-ji}\, [v]_{\tilde{A}_j,\frac{n}{2}} \\
\overset{\gamma>i}{\prec}\ & S^{i+1,\alpha}_{\gamma} + r^{i-|\alpha|} \sum_{|\beta|=i} S^{i+1,\beta}_{-n} + r^{-|\alpha|} \sum_{j=-\infty}^{0} 2^{j(n-i)}\, [v]_{\tilde{A}_j,\frac{n}{2}} + r^{-|\alpha|} \sum_{j=1}^{\infty} 2^{j(-\gamma-|\alpha|)}\, [v]_{\tilde{A}_j,\frac{n}{2}} \\
\overset{i\leq N}{\prec}\ & S^{i+1,\alpha}_{\gamma} + r^{i-|\alpha|} \sum_{|\beta|=i} S^{i+1,\beta}_{-n} + r^{-|\alpha|} \sum_{j=-\infty}^{0} 2^{j(n-N)}\, [v]_{\tilde{A}_j,\frac{n}{2}} + r^{-|\alpha|} \sum_{j=1}^{\infty} 2^{j(-\gamma-|\alpha|)}\, [v]_{\tilde{A}_j,\frac{n}{2}}.
\end{aligned}$$

Consequently, one can prove by induction for $i \in \{0,\ldots,N\}$, that (3.6) holds whenever $\gamma > N$, $|\alpha| \leq i$, for $\tilde{\gamma} := \min(n-N, \gamma-N)$, i.e.

$$S^{i,\alpha}_{\gamma} + S^{i,\alpha}_{-\gamma} \leq C_{\gamma,N} \left(r^{-|\alpha|} \sum_{j=-\infty}^{\infty} 2^{-|j|\tilde{\gamma}}\, [v]_{\tilde{A}_j,\frac{n}{2}} \right),$$

Taking $i = 0$, $\alpha = 0$, we conclude.

Lemma 3.12 □

4 Integrability and Compensation Phenomena: Proof of Theorem 1.4

We will frequently use the following operator

$$H(u,v) := \Delta^{\frac{n}{4}}(uv) - (\Delta^{\frac{n}{4}}u)v - u\Delta^{\frac{n}{4}}v, \quad \text{for } u,v \in \mathcal{S}(\mathbb{R}^n). \tag{4.1}$$

In general there is no product rule making $H(u,v) \equiv 0$, or $H(u,v)$ an operator of lower order, as would happen if $n \in 4\mathbb{N}$. But in some way this quantity still acts *like* an operator of lower order, as Lemma 4.1 shows.
This was observed in [DLR09]. As remarked there, the compensation phenomena that appear are very similar to the ones in Wente's inequality (see the introduction of [DLR09] for more on that). In fact, in this note we would like to stress that even an argument very similar to Tartar's proof in [Tar85] still works.

It is easy to see that for any $x, y \in \mathbb{R}^n$ and any $p > 0$, $\theta \in [0,1]$ we have for a uniform constant $C_p > 0$

$$||x-y|^p - |y|^p - |x|^p| \leq C_p \begin{cases} |x|^{p\theta}\, |y|^{p(1-\theta)} & \text{if } p \in (0,1], \\ |x|^{p-1}|y| + |x||y|^{p-1} & \text{if } p > 1. \end{cases} \tag{4.2}$$

Consequently,

Lemma 4.1. *For any $u, v \in \mathcal{S}(\mathbb{R}^n)$ we have in the case $n = 1, 2$*

$$|H(u,v)^\wedge| \leq C\, \left|(\Delta^{\frac{n}{8}}u)^\wedge\right| * \left|(\Delta^{\frac{n}{8}}v)^\wedge\right|(\xi),$$

and in the case $n \geq 3$

$$|(H(u,v))^\wedge| \leq C \left|(\Delta^{\frac{n-2}{4}} u)^\wedge\right| * \left|(\Delta^{\frac{1}{2}} v)^\wedge\right| + C\left|(\Delta^{\frac{1}{2}} u)^\wedge\right| * \left|(\Delta^{\frac{n-2}{4}} v)^\wedge\right|.$$

Theorem 4.2. *(Compare to similar results in [Tar85], [DLR09, Theorem 1.2, Theorem 1.3]) Let $u, v \in \mathcal{S}(\mathbb{R}^n)$ and set*

$$H(u,v) := \Delta^{\frac{n}{4}}(uv) - v\Delta^{\frac{n}{4}} u - u\Delta^{\frac{n}{4}} v.$$

Then,

$$\|H(u,v)^\wedge\|_{L^{2,1}(\mathbb{R}^n)} \leq C_n \ \|\Delta^{\frac{n}{4}} u\|_{L^2(\mathbb{R}^n)} \ \|\Delta^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)}.$$

and

$$\|H(u,v)\|_{L^2(\mathbb{R}^n)} \leq C_n \ \|(\Delta^{\frac{n}{4}} u)^\wedge\|_{L^{2,\infty}(\mathbb{R}^n)} \ \|\Delta^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)}.$$

In particular,

$$\|H(u,v)\|_{L^2(\mathbb{R}^n)} \leq C_n \ \|\Delta^{\frac{n}{4}} u\|_{L^2(\mathbb{R}^n)} \ \|\Delta^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)}.$$

***Proof of Theorem* 4.2.**
Lemma 4.1 implies, in the case $n = 1, 2$

$$|(H(u,v))^\wedge| \leq C \left(|\cdot|^{-\frac{n}{4}}\left|(\Delta^{\frac{n}{4}} u)^\wedge\right|\right) * \left(|\cdot|^{-\frac{n}{4}}\left|(\Delta^{\frac{n}{4}} v)^\wedge\right|\right)$$

and in the case $n \geq 3$

$$|(H(u,v))^\wedge| \leq C \left(|\cdot|^{-1}\left|(\Delta^{\frac{n}{4}} u)^\wedge\right|\right) * \left(|\cdot|^{-\frac{n-2}{2}}\left|(\Delta^{\frac{n}{4}} v)^\wedge\right|\right) + C \left(|\cdot|^{-\frac{n-2}{2}}\left|(\Delta^{\frac{n}{4}} u)^\wedge\right|\right) * \left(|\cdot|^{-1}\left|(\Delta^{\frac{n}{4}} v)^\wedge\right|\right).$$

Now we use Hölder's inequality: By Proposition 2.2 we have that

$$\begin{array}{lll} |\cdot|^{-\frac{n}{4}} \in L^{4,\infty}(\mathbb{R}^n), & L^2 \cdot L^{4,\infty} \subset L^{\frac{4}{3},2}, & L^{2,\infty} \cdot L^{4,\infty} \subset L^{\frac{4}{3},\infty}, \\ |\cdot|^{-1} \in L^{n,\infty}(\mathbb{R}^n), & L^2 \cdot L^{n,\infty} \subset L^{\frac{2n}{n+2},2}, & L^{2,\infty} \cdot L^{n,\infty} \subset L^{\frac{2n}{n+2},\infty}, \\ |\cdot|^{-\frac{n-2}{2}} \in L^{\frac{2n}{n-2},\infty}(\mathbb{R}^n), & L^2 \cdot L^{\frac{2n}{n-2},\infty} \subset L^{\frac{n}{n-1},2}, & L^{2,\infty} \cdot L^{\frac{2n}{n-2},\infty} \subset L^{\frac{n}{n-1},\infty}. \end{array}$$

Moreover, convolution acts as follows

$$\begin{array}{ll} L^{\frac{4}{3},2} * L^{\frac{4}{3},2} \subset L^{2,1}, & L^{\frac{4}{3},\infty} * L^{\frac{4}{3},2} \subset L^2, \\ L^{\frac{2n}{n+2},2} * L^{\frac{n}{n-1},2} \subset L^{2,1}, & L^{\frac{2n}{n+2},2} * L^{\frac{n}{n-1},\infty} + L^{\frac{2n}{n+2},\infty} * L^{\frac{n}{n-1},2} \subset L^2. \end{array}$$

We can conclude.

Theorem 4.2 □

5 Localization Results for the Fractional Laplacian

Even though Δ^s is a nonlocal operator, its "differentiating force" concentrates around the point evaluated. Thus, to estimate $\Delta^{\frac{s}{2}}$ at a given point x one has to look "only around" x. In this spirit the following results hold.

5.1 Multiplication with disjoint support

In [DLR09] a special case of the following Lemma is used many times. As a consequence of lower order effects appearing when dealing with dimensions and orders greater than one, we will need it in a more general setting, namely for arbitrary homogeneous multiplier operators.

Lemma 5.1. *Let M be an operator with Fourier multiplier $m \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C})$, $m \in C^\infty(\mathbb{R}^n \backslash \{0\}, \mathbb{C})$, i.e.*

$$Mv := (mv^\wedge)^\vee \quad \text{for any } v \in \mathcal{S}.$$

If m is homogeneous of order $\delta > -n$, for any $a, b \in \mathcal{S}(\mathbb{R}^n, \mathbb{C})$ such that for some $\gamma, d > 0$, $x \in \mathbb{R}^n$, $\operatorname{supp} a \subset B_\gamma(x)$ and $\operatorname{supp} b \subset \mathbb{R}^n \backslash B_{d+\gamma}(x)$,

$$\left|\int_{\mathbb{R}^n} a \ Mb\right| \leq C_M \ d^{-n-\delta} \ \|a\|_{L^1(\mathbb{R}^n)} \ \|b\|_{L^1(\mathbb{R}^n)}.$$

An immediate consequence, taking $m := |\cdot|^{s+t}$, is

Corollary 5.2. *Let $s,t > -n$, $s+t > -n$. Then, for all $a, b \in \mathcal{S}(\mathbb{R}^n, \mathbb{C})$, such that for some $d, \gamma > 0$,* $\operatorname{supp} a \subset B_\gamma(x)$ *and* $\operatorname{supp} b \subset \mathbb{R}^n \backslash B_{d+\gamma}(x)$*,*

$$\left| \int_{\mathbb{R}^n} \Delta^{\frac{s}{2}} a \ \Delta^{\frac{t}{2}} b \right| \leq C_{n,s,t}\ d^{-(n+s+t)}\ \|a\|_{L^1}\ \|b\|_{L^1}.$$

Lemma 5.1 follows from the following proposition, as the commutation of translations and multiplier operators allows us to assume $\operatorname{supp} a \subset B_\gamma(0)$ and $\operatorname{supp} b \subset \mathbb{R}^n \backslash B_{\gamma+d}(0)$.

Proposition 5.3. *Let $m \in C^\infty(\mathbb{R}^n\backslash\{0\}, \mathbb{C}) \cap \mathcal{S}'$. If for some $\delta > -n$ we have that $m(\lambda x) = \lambda^\delta m(x)$ for any $x \in \mathbb{R}^n\backslash\{0\}$ and any $\lambda > 0$,*

$$\left| \int_{\mathbb{R}^n} m\ \varphi^\wedge \right| \leq C_m\ d^{-n-\delta}\ \|\varphi\|_{L^1(\mathbb{R}^n)}, \quad \text{for any } \varphi \in C_0^\infty(\mathbb{R}^n\backslash\overline{B_d(0)}, \mathbb{C}),\ d > 0.$$

Proposition 5.3 again follows from some general facts about the Fourier Transform on tempered distributions:

Proposition 5.4 (Fourier Transform and Homogeneity)**.**

(i) (See [Gra08, Proposition 2.4.8]) Let $f \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C})$ and $f \in C^\infty(\mathbb{R}^n\backslash\{0\}, \mathbb{C})$. If moreover f is weakly homogeneous of order $\delta \in \mathbb{R}$, i.e. $f[\varphi(\lambda \cdot)] = \lambda^{-n-\delta} f[\varphi]$, for all $\varphi \in \mathcal{S}(\mathbb{R}^n, \mathbb{C})$, then $f^\wedge, f^\vee \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C})$ also belong to $C^\infty(\mathbb{R}^n\backslash\{0\}, \mathbb{C})$.

(ii) Let $f \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C})$. If f is weakly homogeneous of order $\delta \in \mathbb{R}$, then $f^\wedge \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C})$ and $f^\vee \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C})$ are weakly homogeneous of order $\gamma = -n - \delta$.

(iii) Let $g \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C})$, $g \in C^\infty(\mathbb{R}^n\backslash\{0\}, \mathbb{C})$. If g is weakly homogeneous of order γ, then also pointwise $g(\lambda x) = \lambda^\gamma g(x)$, for every $x \in \mathbb{R}^n\backslash\{0\}$, $\lambda > 0$.

(iv) Let $g \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C})$, $g \in C^\infty(\mathbb{R}^n\backslash\{0\}, \mathbb{C})$. If there is $\gamma \leq 0$ such that $g(\lambda x) = \lambda^\gamma g(x)$ for every $x \in \mathbb{R}^n\backslash\{0\}$, $\lambda > 0$ then

$$\left| \int g\ \varphi \right| \leq d^\gamma \|g\|_{L^\infty(\mathbb{S}^{n-1})}\ \|\varphi\|_{L^1(\mathbb{R}^n)}, \quad \text{for every } \varphi \in C_0^\infty(\mathbb{R}^n\backslash\overline{B_d(0)}),\ d > 0.$$

5.2 Equations with disjoint support localize

As a consequence of Corollary 5.2 we can *de facto* localize our equations, i.e. replace multiplications of nonlocal operators applied to mappings with disjoint support (which would be zero in the case of local operators) by an operator of order zero:

Lemma 5.5 (Localizing)**.** *Let $b \in H^{\frac{n}{2}}(\mathbb{R}^n)$. Assume there is $d, \gamma > 0$, $x \in \mathbb{R}^n$ such that for $E := B_{\gamma+d}(x)$,* $\operatorname{supp} b \subset \mathbb{R}^n\backslash E$*. Then there is a function $a \in L^2(\mathbb{R}^n)$ such that for $D := B_\gamma(x)$*

$$\int_{\mathbb{R}^n} \Delta^{\frac{n}{4}} b\ \Delta^{\frac{n}{4}} \varphi = \int_{\mathbb{R}^n} a\ \varphi, \quad \text{for every } \varphi \in C_0^\infty(D)$$

and $\|a\|_{L^2(\mathbb{R}^n)} \leq C_{D,E} \|b\|_{L^2(\mathbb{R}^n)}$.

Proof of Lemma 5.5.
We are going to show that

$$|f(\varphi)| := \left| \int_{\mathbb{R}^n} \Delta^{\frac{n}{4}} b\ \Delta^{\frac{n}{4}} \varphi \right| \leq C_{D,E} \|\varphi\|_{L^2(\mathbb{R}^n)} \quad \text{for every } \varphi \in C_0^\infty(D). \tag{5.1}$$

Then $f(\cdot)$ is a linear and bounded operator on the dense subspace $C_0^\infty(D) \subset L^2(D)$. Hence, it is extendable to all of $L^2(D)$. Being a linear functional, by Riesz' representation theorem there exists $a \in L^2(D)$ such that $f(\varphi) = \langle a, \varphi \rangle_{L^2(D)}$ for every $\varphi \in L^2(D)$.

It remains to prove (5.1), which is done as in the proofs of [DLR09]. Set $r := \frac{1}{2}(\gamma + d)$, so that $E = B_{2r}(x) \supset D$. Applying Corollary 5.2

$$\int_{\mathbb{R}^n} \Delta^{\frac{n}{4}} b \; \Delta^{\frac{n}{4}} \varphi = \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} \Delta^{\frac{n}{4}} (\eta^k_{r,x} b) \; \Delta^{\frac{n}{4}} \varphi =: \sum_{k=1}^{\infty} I_k.$$

If $k \geq 3$, using that the support of η^k_r and φ are disjoint, more precisely by Corollary 5.2,

$$II_k \overset{C.5.2}{\prec} 2^{-2kn} \|\eta^k_r b\|_{L^1(\mathbb{R}^n)} \|\varphi\|_{L^1(\mathbb{R}^n)} \prec 2^{-\frac{3}{2}kn} \|\eta^k_r b\|_{L^2(\mathbb{R}^n)} \|\varphi\|_{L^1(\mathbb{R}^n)} \prec 2^{-\frac{3}{2}kn} \|b\|_{L^2(\mathbb{R}^n)} \|\varphi\|_{L^2(D)}.$$

For $1 \leq k \leq 3$ we use that the support of a and φ are disjoint, to get also by Corollary 5.2

$$II_k \prec d^{-32n} \|b\|_{L^2(\mathbb{R}^n)} \|\varphi\|_{L^2(D)}.$$

Consequently,

$$\sum_{k=1}^{\infty} II_k \leq C_{D,E} \|b\|_{L^2(\mathbb{R}^n)} \; \|\varphi\|_{L^2(D)}.$$

Lemma 5.5 □

5.3 Hodge decomposition and Local Estimates of s-harmonic Functions: Proof of Theorem 1.6

If for an integrable function h we have weakly $\Delta h = 0$ in a, say, big ball, we can estimate $\|h\|_{L^2(B_r)} \leq C \left(\frac{r}{\rho}\right)^2 \|h\|_{L^2(B_\rho)}$, for $0 < r < \rho$. The goal of this subsection is to prove in Lemma 5.8 a similar estimate, for the nonlocal operator $\Delta^{\frac{n}{4}}$.

Proposition 5.6. *Let $s \in (0, \frac{n}{2})$. Then for any $x \in \mathbb{R}^n$, $r > 0$ and $v \in \mathcal{S}$, such that $\operatorname{supp} v \subset B_r(x)$, and any $k \in \mathbb{N}_0$,*

$$\left\| \left| (\Delta^{\frac{s}{2}} \eta^k_{r,x})^\wedge \right| * \left| (\Delta^{-\frac{s}{2}} v)^\wedge \right| \right\|_{L^2(\mathbb{R}^n)} \leq C_s 2^{-ks} \|v\|_{L^2(\mathbb{R}^n)}.$$

***Proof of Proposition* 5.6.**
By convolution rule we have

$$\left\| \left| (\Delta^{\frac{s}{2}} \eta^k_{r,x})^\wedge \right| * \left| (\Delta^{-\frac{s}{2}} v)^\wedge \right| \right\|_{L^2(\mathbb{R}^n)} \prec \|(\Delta^{\frac{s}{2}} \eta^k_{r,x})^\wedge\|_{L^1(\mathbb{R}^n)} \; \|(\Delta^{-\frac{s}{2}} v)^\wedge\|_{L^2(\mathbb{R}^n)}. \tag{5.2}$$

By Lemma 2.8,

$$\|(\Delta^{-\frac{s}{2}} v)^\wedge\|_{L^2(\mathbb{R}^n)} = \|\Delta^{-\frac{s}{2}} v\|_{L^2(\mathbb{R}^n)} \leq C_s r^s \|v\|_{L^2(\mathbb{R}^n)}. \tag{5.3}$$

Furthermore, Proposition 2.13 implies

$$\|(\Delta^{\frac{s}{2}} \eta^k_{r,x})^\wedge\|_{L^1(\mathbb{R}^n)} \leq C_s (2^k r)^{-s}. \tag{5.4}$$

Together, (5.2), (5.3) and (5.4) give the claim.

Proposition 5.6 □

As a consequence we have

Proposition 5.7. *There is a uniform constant $C > 0$ such that for any $r > 0$, $x \in \mathbb{R}^n$, $v \in \mathcal{S}$, such that $\operatorname{supp} v \subset B_r(x)$, and for any $k \in \mathbb{N}_0$*

$$\|\Delta^{\frac{n}{4}} (\eta^k_{r,x} \Delta^{-\frac{n}{4}} v)\|_{L^2(\mathbb{R}^n)} \leq C \; 2^{-k\frac{1}{4}} \|v\|_{L^2(\mathbb{R}^n)}.$$

***Proof of Proposition* 5.7.**
We have according to (4.1) $\Delta^{\frac{n}{4}} (\eta^k_{r,x} \Delta^{-\frac{n}{4}} v) = (\Delta^{\frac{n}{4}} \eta^k_{r,x}) \Delta^{-\frac{n}{4}} v + \eta^k_{r,x} v + H(\eta^k_{r,x}, \Delta^{-\frac{n}{4}} v)$. By the support condition on v for $k \geq 1$ we have $\eta^k_{r,x} v = 0$ so trivially for any $k \in \mathbb{N}_0$, $\|\eta^k_{r,x} v\|_{L^2(\mathbb{R}^n)} \leq 2^{\frac{n}{2}} \; 2^{-k\frac{n}{4}} \|v\|_{L^2(\mathbb{R}^n)}$. Next, applying Proposition 2.13 for $s = \frac{n}{2}$ and $p = 4$ and Lemma 2.8 for $s = \frac{n}{2}$ and $p' = 4$, we have

$$\|(\Delta^{\frac{n}{4}} \eta^k_{r,x}) \Delta^{-\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)} \leq \|(\Delta^{\frac{n}{4}} \eta^k_{r,x})\|_{L^4} \; \|\Delta^{-\frac{n}{4}} v\|_{L^4} \prec 2^{-k\frac{n}{4}} r^{-\frac{n}{4}} r^{\frac{n}{4}} \; \|v\|_{L^2}.$$

Thus, we have shown that

$$\|\Delta^{\frac{n}{4}} (\eta^k_{r,x} \Delta^{-\frac{n}{4}} v)\|_{L^2(\mathbb{R}^n)} \prec 2^{-k\frac{n}{4}} \|v\|_{L^2(\mathbb{R}^n)} + \|H(\eta^k_{r,x}, \Delta^{-\frac{n}{4}} v)\|_{L^2(\mathbb{R}^n)}. \tag{5.5}$$

By Lemma 4.1 we have that in the case $n = 1, 2$

$$\|H(\eta^k_{r,x}, \Delta^{-\frac{n}{4}} v)\|_{L^2(\mathbb{R}^n)} \prec \| \big|(\Delta^{\frac{n}{8}} \eta^k_{r,x})^\wedge\big| * \big|(\Delta^{-\frac{n}{8}} v)^\wedge\big| \|_{L^2(\mathbb{R}^n)},$$

and in the case $n \geq 3$

$$\|H(\eta^k_{r,x}, \Delta^{-\frac{n}{4}} v)\|_{L^2(\mathbb{R}^n)} \prec \| \Big|(\Delta^{\frac{n-2}{4}} \eta^k_{r,x})^\wedge\Big| * \Big|(\Delta^{\frac{2-n}{4}} v)^\wedge\Big| \|_{L^2} + \| \Big|(\Delta^{\frac{1}{2}} \eta^k_{r,x})^\wedge\Big| * \Big|(\Delta^{-\frac{1}{2}} v)^\wedge\Big| \|_{L^2}.$$

That is, in order to prove the claim we need the estimate

$$\| \big|(\Delta^{\frac{s}{2}} \eta^k_{r,x})^\wedge\big| * \big|(\Delta^{-\frac{s}{2}} v)^\wedge\big| \|_{L^2} \leq C_s\ 2^{-ks} \|v\|_{L^2} \tag{5.6}$$

where $s = \frac{n}{4}$ in the case $n = 1, 2$ and $s = \frac{n-2}{2}$ or $s = 1$ in the case $n \geq 3$. In all three cases we have that $0 < s < \frac{n}{2}$ and Proposition 5.6 implies (5.6). Plugging these estimates into (5.5) we conclude.

Proposition 5.7 □

Lemma 5.8 (Estimate of the Harmonic Term)**.** *Let $h \in L^2(\mathbb{R}^n)$, such that*

$$\int_{\mathbb{R}^n} h\ \Delta^{\frac{n}{4}} \varphi = 0 \quad \text{for any } \varphi \in C_0^\infty(B_{\Lambda r}(x)). \tag{5.7}$$

for some $\Lambda > 0$. Then, for a uniform constant $C > 0$, $\|h\|_{L^2(B_r(x))} \leq C\ \Lambda^{-\frac{1}{4}} \|h\|_{L^2(\mathbb{R}^n)}$.

***Proof of Lemma* 5.8.**
It suffices to prove the claim for large Λ, say $\Lambda \geq 8$. Let $k_0 \in \mathbb{N}$, $k_0 \geq 3$, such that $\Lambda < 2^{k_0} \leq 2\Lambda$. Approximate h by functions $h_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ such that for any $\varepsilon > 0$ the distance $\|h - h_\varepsilon\|_{L^2(\mathbb{R}^n)} \leq \varepsilon$ and $\|h_\varepsilon\|_{L^2(\mathbb{R}^n)} \leq 2\|h\|_{L^2(\mathbb{R}^n)}$. By Riesz' representation theorem, $\|h_\varepsilon\|_{L^2(B_r(x))} = \sup_v \int h_\varepsilon v$, where the supremum is over all $v \in C_0^\infty(B_r(x))$ such that $\|v\|_{L^2} \leq 1$. For such a v, by Proposition 5.7

$$\|\Delta^{\frac{n}{4}} (\eta^k_{r,x} \Delta^{-\frac{n}{4}} v)\|_{L^2(\mathbb{R}^n)} \leq C\ 2^{-\frac{k}{4}}. \tag{5.8}$$

In order to apply (5.7), we rewrite

$$\int h_\varepsilon\ v \quad = \quad \sum_{k=k_0-1}^{\infty} \int h_\varepsilon\ \Delta^{\frac{n}{4}} (\eta^k_{r,x} \Delta^{-\frac{n}{4}} v) + \sum_{k=0}^{k_0-2} \int h_\varepsilon\ \Delta^{\frac{n}{4}} (\eta^k_{r,x} \Delta^{-\frac{n}{4}} v) =: I + II.$$

The second term II goes to zero as $\varepsilon \to 0$. In fact, for $k \leq k_0 - 2$ we have that $\operatorname{supp} \eta^k_{r,x} \subset B_{\Lambda r}(x)$ and thus

$$\int_{\mathbb{R}^n} h_\varepsilon\ \Delta^{\frac{n}{4}} (\eta^k_{r,x} \Delta^{-\frac{n}{4}} v) \overset{(5.7)}{=} \int (h_\varepsilon - h)\ \Delta^{\frac{n}{4}} (\eta^k_{r,x} \Delta^{-\frac{n}{4}} v) \leq \varepsilon\ \|\Delta^{\frac{n}{4}} (\eta^k_{r,x} \Delta^{-\frac{n}{4}} v)\|_{L^2(\mathbb{R}^n)} \overset{(5.8)}{\leq} C_\Lambda\ \varepsilon.$$

For the remaining term I we have, using again Proposition 5.7,

$$I \overset{(5.8)}{\leq} \|h\|_{L^2(\mathbb{R}^n)} \sum_{k=k_0-1}^{\infty} 2^{-\frac{k}{4}}.$$

We arrive at $\int h_\varepsilon\ v \leq C\varepsilon + C\Lambda^{-\frac{1}{4}}\ \|h\|_{L^2(\mathbb{R}^n)}$, which converges to the claim if $\varepsilon \to 0$.

Lemma 5.8 □

Now we are able to prove Theorem 1.6.
***Proof of Theorem* 1.6.**
As usual we have $\|v\|_{L^2(B_r(x))} = \sup_f \int v\ f$, where the supremum is taken over all $f \in L^2(\mathbb{R}^n)$ such that $\|f\|_{L^2} \leq 1$. By Lemma 2.9 and Lemma 5.8, we decompose $f = \Delta^{\frac{n}{4}} \varphi + h$, $\varphi \in H^{\frac{n}{2}}(\mathbb{R}^n)$ and $\operatorname{supp} \varphi \subset B_{\Lambda r}(x)$, $\|h\|_{L^2(B_r(x))} \leq C\ \Lambda^{-\frac{1}{4}}$ for arbitrarily large $\Lambda > 0$. Thus, by the support condition on v,

$$\|v\|_{L^2(B_r(x))} \leq C \sup_{\substack{\varphi \in C_0^\infty(B_{\Lambda r}(x)) \\ \|\Delta^{\frac{n}{4}} \varphi\|_{L^2(\mathbb{R}^n)} \leq 1}} \int v \Delta^{\frac{n}{4}} \varphi + C\Lambda^{-\frac{1}{4}}\ \|v\|_{L^2(B_r(x))}.$$

Taking Λ large enough, we can absorb and conclude.

Theorem 1.6 □

5.4 Products of lower order operators localize well

The goal of this subsection are Lemma 5.10 and Lemma 5.11, which essentially state that terms of the form $\Delta^{\frac{s}{2}} a\, \Delta^{\frac{n}{4}-\frac{s}{2}} b$ "localize alright", if s is neither of the extremal values 0 nor $\frac{n}{2}$.

Proposition 5.9 (Lower Order Operators and L^2)**.** *For any $s \in (0, \frac{n}{2})$, M_1, M_2 zero multiplier operators there exists a constant $C_{M_1,M_2,s} > 0$ such that for any $u, v \in \mathcal{S}$,*

$$\|M_1 \Delta^{\frac{2s-n}{4}} u\ M_2 \Delta^{-\frac{s}{2}} v\|_{L^2(\mathbb{R}^n)} \leq C_{M_1,M_2,s} \|u\|_{L^2(\mathbb{R}^n)}\ \|v\|_{L^2(\mathbb{R}^n)}.$$

***Proof of Proposition* 5.9.**
Set $p := \frac{n}{s}$ and $q := \frac{2n}{n-2s}$. As $2 < p, q < \infty$ (using also Hörmander's multiplier theorem, [Hör60]),

$$\begin{aligned}\|M_1 \Delta^{\frac{2s-n}{4}} u\ M_2 \Delta^{-\frac{s}{2}} v\|_{L^2} &\overset{p,q\in(1,\infty)}{\prec} \|\Delta^{\frac{2s-n}{4}} u\|_{L^p}\ \|\Delta^{-\frac{s}{2}} v\|_{L^q} \\ &\overset{\substack{p,q\in[2,\infty)\\ P.2.2}}{\prec} \||\cdot|^{\frac{2s-n}{2}} u^\wedge\|_{L^{p'},2}\ \||\cdot|^{-s}\ v^\wedge\|_{L^{q'},2} \overset{P.2.2}{\prec} \|u\|_{L^2}\ \|v\|_{L^2}.\end{aligned}$$

Proposition 5.9 □

Lemma 5.10. *Let $s \in (0, \frac{n}{2})$ and M_1, M_2 zero multiplier operators. Then there is a constant $C_{M_1,M_2,s} > 0$ such that the following holds. For any $u, v \in \mathcal{S}$ and any $\Lambda > 2$,*

$$\|M_1 \Delta^{\frac{s}{2}} u\ M_2 \Delta^{\frac{n}{4}-\frac{s}{2}} v\|_{L^2(B_r(x))} \leq C_{M_1,M_2,s} \left(\|\Delta^{\frac{n}{4}} u\|_{L^2(B_{2\Lambda r}(x))} + \Lambda^{-s} \sum_{k=1}^{\infty} 2^{-ks} \|\eta^k_{\Lambda r,x} \Delta^{\frac{n}{4}} u\|_{L^2} \right) \|\Delta^{\frac{n}{4}} v\|_{L^2}.$$

***Proof of Lemma* 5.10.**
As usual

$$\|\Delta^{\frac{s}{2}} M_1 u\ \Delta^{\frac{n}{4}-\frac{s}{2}}\ M_2 v\|_{L^2(B_r(x))} = \sup_{\substack{\varphi \in C_0^\infty(B_r(x),\mathbb{C})\\ \|\varphi\|_{L^2}\leq 1}} \left| \int M_1 \Delta^{\frac{s}{2}} u\ M_2 \Delta^{\frac{n}{4}-\frac{s}{2}}\ v\ \varphi \right|.$$

For such a φ we then decompose $\Delta^{\frac{s}{2}} u$ into the part which is close to $B_r(x)$ and the far-off part:

$$\begin{aligned} & \int M_1 \Delta^{\frac{s}{2}} u\ M_2 \Delta^{\frac{n}{4}-\frac{s}{2}}\ v\ \varphi \\ = & \int M_1 \Delta^{\frac{s}{2}-\frac{n}{4}} (\eta_{\Lambda r} \Delta^{\frac{n}{4}} u)\ M_2 \Delta^{\frac{n}{4}-\frac{s}{2}}\ v\ \varphi + \sum_{k=1}^{\infty} \int M_1 \Delta^{\frac{s}{2}-\frac{n}{4}} (\eta^k_{\Lambda r} \Delta^{\frac{n}{4}} u)\ M_2 \Delta^{-\frac{s}{2}} \Delta^{\frac{n}{4}}\ v\ \varphi \\ =: & \ I + \sum_{k=1}^{\infty} II_k. \end{aligned}$$

We first estimate the I by Proposition 5.9

$$|I| \prec \|\eta_{\Lambda r} \Delta^{\frac{n}{4}} u\|_{L^2}\ \|\Delta^{\frac{n}{4}} v\|_{L^2}.$$

In order to estimate II_k, observe that for any $\varphi \in C_0^\infty(B_r(x), \mathbb{C})$, $\|\varphi\|_{L^2} \leq 1$, $s \in (0, \frac{n}{2})$, if we set $p := \frac{2n}{n+2s} \in (1, 2)$

$$\begin{aligned} \|\varphi\ M_2 \Delta^{-\frac{s}{2}} \Delta^{\frac{n}{4}} v\|_{L^1} &\prec r^s\ \|\Delta^{-\frac{s}{2}} \Delta^{\frac{n}{4}} v\|_{L^{p'}(\mathbb{R}^n)} \overset{p'\geq 2}{\prec} r^s\ \||\cdot|^{-s} (\Delta^{\frac{n}{4}} v)^\wedge\|_{L^{p,2}(\mathbb{R}^n)} \\ &\prec r^s\ \||\cdot|^{-s}\|_{L^{\frac{n}{s},\infty}}\ \|(\Delta^{\frac{n}{4}} v)^\wedge\|_{L^2} \prec r^s\ \|\Delta^{\frac{n}{4}} v\|_{L^2}. \end{aligned} \tag{5.9}$$

Hence, as for any $k \geq 1$ we have $\operatorname{dist}(\operatorname{supp} \varphi, \operatorname{supp} \eta^k_{\Lambda r}) \succ 2^k \Lambda r$,

$$\begin{aligned} \left| \int M_1 \Delta^{\frac{s}{2}-\frac{n}{4}} (\eta^k_{\Lambda r} \Delta^{\frac{n}{4}} u)\ M_2 \Delta^{\frac{n}{4}-\frac{s}{2}}\ v\ \varphi \right| &\overset{L.5.1}{\prec} (2^k \Lambda r)^{-n-s+\frac{n}{2}} \|\eta^k_{\Lambda r} \Delta^{\frac{n}{4}} u\|_{L^1}\ \|M_2 \Delta^{\frac{n}{4}-\frac{s}{2}}\ v\ \varphi\|_{L^1} \\ &\overset{(5.9)}{\prec} 2^{-ks} \Lambda^{-s} \|\eta^k_{\Lambda r} \Delta^{\frac{n}{4}} u\|_{L^2}\ \|\Delta^{\frac{n}{4}} v\|_{L^2}. \end{aligned}$$

Lemma 5.10 □

By a similar argument, one can prove the following Lemma.

Lemma 5.11. *Let $s \in (0, \frac{n}{2})$ and M_1, M_2 be zero-multiplier operators. Then there is a constant $C_{M_1,M_2,s} > 0$ such that the following holds. For any $u, v \in \mathcal{S}$ and for any $\Lambda > 2$, $r > 0$, $B_r \equiv B_r(x) \subset \mathbb{R}^n$,*

$$\begin{aligned} & \|M_1 \Delta^{\frac{s}{2}} u \ M_2 \Delta^{\frac{n}{4} - \frac{s}{2}} \ v\|_{L^2(B_r(x))} \\ \leq \ & C_{M_1,M_2,s} \left(\|\eta_{\Lambda r,x} \Delta^{\frac{n}{4}} u\|_{L^2} \ \|\eta_{\Lambda r,x} \Delta^{\frac{n}{4}} v\|_{L^2} + \Lambda^{-s} \ \|\eta_{\Lambda r,x} \Delta^{\frac{n}{4}} v\|_{L^2} \sum_{k=1}^{\infty} 2^{-sk} \|\eta^k_{\Lambda r,x} \Delta^{\frac{n}{4}} u\|_{L^2} \right) \\ & + C_{M_1,M_2,s} \ \Lambda^{s-\frac{n}{2}} \ \|\eta_{\Lambda r,x} \Delta^{\frac{n}{4}} u\|_{L^2} \sum_{l=1}^{\infty} 2^{(s-\frac{n}{2})l} \|\eta^l_{\Lambda r,x} \Delta^{\frac{n}{4}} v\|_{L^2} \\ & + C_{M_1,M_2,s} \ \Lambda^{-\frac{n}{2}} \sum_{k,l=1}^{\infty} 2^{-(ks+l(\frac{n}{2}-s))} \|\eta^k_{\Lambda r,x} \Delta^{\frac{n}{4}} u\|_{L^2} \ \|\eta^l_{\Lambda r,x} \Delta^{\frac{n}{4}} v\|_{L^2}. \end{aligned}$$

5.5 Fractional Product Rules for Polynomials

It is obvious, that for any constant $c \in \mathbb{R}$ and any $\varphi \in \mathcal{S}$, $s > 0$, $\Delta^{\frac{s}{2}}(c\varphi) = c\Delta^{\frac{s}{2}}\varphi$. In this section, we are going to extend this kind of product rule to polynomials of degree greater than zero, which in our application will be mean value polynomials as in (3.1). As we have to deal with dimensions greater than one, our mean value polynomials will be in general also of arbitrary degree, making such calculations necessary.

Proposition 5.12 (Product Rule for Polynomials)**.** *Let $N \in \mathbb{N}_0$, $s \geq N$. Then for any multiplier operator M defined by*

$$(Mv)^\wedge = mv^\wedge, \quad \textit{for any } v \in \mathcal{S},$$

for $m \in C^\infty(\mathbb{R}^n \backslash \{0\}, \mathbb{C})$ and homogeneous of order zero, there exists for every multiindex $\beta \in (\mathbb{N}_0)^n$, $|\beta| \leq N$, a multiplier operator $M_\beta \equiv M_{\beta,s,N}$, $M_\beta = M$ if $|\beta| = 0$, with multiplier $m_\beta \in C^\infty(\mathbb{R}^n \backslash \{0\}, \mathbb{C})$ also homogeneous of order zero such that the following holds. Let $Q = x^\alpha$ for some multiindex $\alpha \in (\mathbb{N}_0)^n$, $|\alpha| \leq N$. Then

$$M\Delta^{\frac{s}{2}}(Q\varphi) = \sum_{|\beta| \leq |\alpha|} \partial^\beta Q \ M_\beta \Delta^{\frac{s-|\beta|}{2}} \varphi \quad \textit{for any } \varphi \in \mathcal{S}. \tag{5.10}$$

Consequently, for any polynomial $P = \sum_{|\alpha| \leq N} c_\alpha x^\alpha$,

$$M\Delta^{\frac{s}{2}}(P\varphi) = \sum_{|\beta| \leq N} \partial^\beta P \ M_\beta \Delta^{\frac{s-|\beta|}{2}} \varphi \quad \textit{for any } \varphi \in \mathcal{S}.$$

***Proof of Proposition* 5.12.**
The claim for P follows immediately from the claim about Q as left- and right-hand side are linear in the space of polynomials. We will prove the claim for Q by induction on N, but first we make some preperatory observations. For an operator M with multiplier m as requested, for $\alpha \in (\mathbb{N}_0)^n$ a multiindex and $s \in \mathbb{R}$ set

$$m_{\alpha,s}(\xi) := \frac{1}{(2\pi \mathbf{i})^{|\alpha|}} |\xi|^{|\alpha|-s} \ \partial^\alpha (|\xi|^s \ m(\xi)), \quad \xi \in \mathbb{R}^n \backslash \{0\},$$

and let $M_{\alpha,s}$ be the according operator with $m_{\alpha,s}$ as Fourier multiplier. In a slight abuse of this notation, for multiindices with only one entry we will write

$$M_{k,s} \equiv M_{\alpha_k,s} \quad \text{for } k \in (1, \ldots, n),$$

where $\alpha_k = (0, \ldots, 0, 1, 0 \ldots, 0)$ and the 1 is exactly at the kth entry of α_k. Note that $m_{\alpha,s}(\cdot)$ is homogeneous of order zero. Also, we have the following relation for any $s \in \mathbb{R}$,

$$(M_{\alpha,s})_{\beta,s-|\alpha|} = M_{\alpha+\beta,s}. \tag{5.11}$$

Observe furthermore that

$$x_1 v(x) = -\frac{1}{2\pi \mathbf{i}} (\partial_1 v^\wedge)^\vee (x),$$

so for $s \geq 1$

$$\left(M\Delta^{\frac{s}{2}}((\cdot)_1 v)\right)^\wedge(\xi) = -\frac{1}{2\pi\mathbf{i}}\,\partial_1(M\Delta^{\frac{s}{2}}v)^\wedge(\xi) + \frac{1}{2\pi\mathbf{i}}\partial_1(m(\xi)|\xi|^s)\; v^\wedge(\xi),$$

that is

$$M\Delta^{\frac{s}{2}}((\cdot)_1 v)(x) = x_1 M\Delta^{\frac{s}{2}}v + M_{1,s}\Delta^{\frac{s-1}{2}}v. \tag{5.12}$$

So one could suspect that for $Q = x^\alpha$ for some multiindex α, $|\alpha| \leq s$,

$$M\Delta^{\frac{s}{2}}(Q\varphi) = \sum_{|\beta|\leq s} \partial^\beta Q\, \frac{1}{\beta!}M_{\beta,s}\,\Delta^{\frac{s-|\beta|}{2}}\varphi. \tag{5.13}$$

where $\beta! := \beta_1!\dots\beta_n!$. This is of course true if $Q \equiv 1$. As induction hypothesis, fix $N > 0$ and assume (5.13) to be true for any monomial $\tilde{Q}$ of degree at most $\tilde{N} < N$ whenever $s \geq \tilde{N}$ and M is an operator with the desired properties. Let then Q be a monomial of degree at most N, and assume $s \geq N$. We decompose w.l.o.g. $Q = x_1\tilde{Q}$ for some monomial $\tilde{Q}$ of degree at most $N-1$. Then,

$$M\Delta^{\frac{s}{2}}(Q\varphi) \overset{(5.12)}{=} x_1 M\Delta^{\frac{s}{2}}\left(\tilde{Q}\varphi\right) + M_{1,s}\Delta^{\frac{s-1}{2}}\left(\tilde{Q}\varphi\right). \tag{5.14}$$

For a multiindex $\beta = (\beta_1,\dots,\beta_n) \in (\mathbb{N}_0)^n$ let us set

$$\tau_1(\beta) := (\beta_1+1,\beta_2,\dots,\beta_n) \quad \text{and} \quad \tau_{-1}(\beta) := (\beta_1-1,\beta_2,\dots,\beta_n).$$

Observe that

$$\partial^\beta(x_1 Q) = \beta_1\partial^{\tau_{-1}(\beta)}Q + x_1\partial^\beta Q. \tag{5.15}$$

Applying now in (5.14) the induction hypothesis (5.13) on $M\Delta^{\frac{s}{2}}$ and $M_{1,s}\Delta^{\frac{s-1}{2}}$, we have

$$M\Delta^{\frac{s}{2}}(Q\varphi) \quad \overset{(5.11)}{=} \quad \sum_{|\beta|\leq s} x_1\partial^\beta\tilde{Q}\,\frac{1}{\beta!}M_{\beta,s}\,\Delta^{\frac{s-|\beta|}{2}}\varphi + \sum_{|\tilde{\beta}|\leq s-1}\partial^{\tilde{\beta}}\tilde{Q}\,\frac{1}{\tilde{\beta}!}\left(M_{\tau_1(\tilde{\beta}),s}\right)\,\Delta^{\frac{s-|\tau_1(\tilde{\beta})|}{2}}\varphi.$$

Next, by (5.15)

$$= \sum_{|\beta|\leq s}\partial^\beta\left(x_1\tilde{Q}\right)\,\frac{1}{\beta!}M_{\beta,s}\,\Delta^{\frac{s-|\beta|}{2}}\varphi - \sum_{\substack{|\beta|\leq s\\ \beta_1\geq 1}}\partial^{\tau_{-1}(\beta)}\tilde{Q}\,\frac{\beta_1}{\beta!}M_{\beta,s}\,\Delta^{\frac{s-|\beta|}{2}}\varphi + \sum_{|\tilde{\beta}|\leq s-1}\partial^{\tilde{\beta}}\tilde{Q}\,\frac{1}{\tilde{\beta}!}\,M_{\tau_1(\tilde{\beta}),s}\,\Delta^{\frac{s-|\tau_1(\tilde{\beta})|}{2}}\varphi$$

$$= \sum_{|\beta|\leq s}\partial^\beta\left(x_1\tilde{Q}\right)\,\frac{1}{\beta!}M_{\beta,s}\,\Delta^{\frac{s-|\beta|}{2}}\varphi.$$

Proposition 5.12 □

Proposition 5.13. *There is a uniform constant $C > 0$ such that the following holds: Let $u \in \mathcal{S}$ and P any polynomial of degree at most $N := \lceil\frac{n}{2}\rceil - 1$. Then for any $\Lambda > 2$, $B_r(x_0) \subset \mathbb{R}^n$, $\varphi \in C_0^\infty(B_r(x_0))$, $\|\Delta^{\frac{n}{4}}\varphi\|_{L^2(\mathbb{R}^n)} \leq 1$,*

$$\begin{aligned}
&\|\Delta^{\frac{n}{4}}(P\varphi) - P\Delta^{\frac{n}{4}}\varphi\|_{L^2(B_r(x_0))}\\
\leq\;& C\left(\|\Delta^{\frac{n}{4}}(\eta_{\Lambda r,x_0}(u-P))\|_{L^2(\mathbb{R}^n)} + \|\Delta^{\frac{n}{4}}u\|_{L^2(B_{2\Lambda r}(x_0))} + \Lambda^{-1}\sum_{k=1}^{\infty}2^{-k}\|\eta^k_{\Lambda r,x_0}\Delta^{\frac{n}{4}}u\|_{L^2(\mathbb{R}^n)}\right).
\end{aligned}$$

***Proof of Proposition* 5.13.**
By Proposition 5.12 (where we take M the identity and $s = \frac{n}{2}$) $\Delta^{\frac{n}{4}}(P\varphi) - P\Delta^{\frac{n}{4}}\varphi = \sum\limits_{1\leq|\beta|\leq N}\partial^\beta P\; M_\beta\Delta^{\frac{n-2|\beta|}{4}}\varphi$. As we estimate the L^2-norm on B_r and there $\eta_{\Lambda r} \equiv 1$, we will further rewrite

$$\begin{aligned}
&= \; -\sum_{1\leq|\beta|\leq N}\partial^\beta(\eta_{\Lambda r}(u-P))M_\beta\Delta^{\frac{n-2|\beta|}{4}}\varphi + \sum_{1\leq|\beta|\leq N}\partial^\beta u\; M_\beta\Delta^{\frac{n-2|\beta|}{4}}\varphi\\
&=: \; \sum_{1\leq|\beta|\leq N}(I_\beta + II_\beta) \qquad \text{on } B_r(x_0).
\end{aligned}$$

As $1 \leq |\beta| \leq N < \frac{n}{2}$, we have by Lemma 5.10 for $v = \varphi$

$$\|II_\beta\|_{L^2(B_r)} \prec\prec \|\Delta^{\frac{n}{4}} u\|_{L^2(B_{2\Lambda r})} + \Lambda^{-1} \sum_{k=1}^{\infty} 2^{-k} \|\eta^k_{\Lambda r} \Delta^{\frac{n}{4}} u\|_{L^2}.$$

We can write

$$I_\beta = M_\beta \Delta^{\frac{2|\beta|-n}{4}} \ \Delta^{\frac{n}{4}}(\eta_{\Lambda r}(v-P)) \ M_\beta \Delta^{-\frac{|\beta|}{2}} \Delta^{\frac{n}{4}} \varphi$$

and by Proposition 5.9 applied to $\Delta^{\frac{n}{4}}(\eta_{\Lambda r}(u-P))$ and $\Delta^{\frac{n}{4}}\varphi$ for $s = |\beta|$

$$\|I_\beta\|_{L^2(\mathbb{R}^n)} \prec \|\Delta^{\frac{n}{4}}(\eta_{\Lambda r}(u-P))\|_{L^2(\mathbb{R}^n)}.$$

Proposition 5.13 $\square$

6 Local Estimates and Compensation: Proof of Theorem 1.5

Theorem 1.5 is essentially a consequence of the following two results.

Lemma 6.1. *There is a uniform constant $C > 0$ such that for any ball $B_r(x_0) \subset \mathbb{R}^n$, $\varphi \in C_0^\infty(B_r(x_0))$, $\|\Delta^{\frac{n}{4}}\varphi\|_{L^2} \leq 1$, and $\Lambda > 4$ as well as for any $v \in \mathcal{S}(\mathbb{R}^n)$,*

$$\|H(v,\varphi)\|_{L^2(B_r(x_0))} \leq C \ \left([v]_{B_{4\Lambda r}(x_0),\frac{n}{4}} + \|\Delta^{\frac{n}{4}} v\|_{B_{2\Lambda r}(x_0)} + \Lambda^{-\frac{1}{2}} \|\Delta^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)} \right).$$

***Proof of Lemma* 6.1.**
We have for almost every point in $B_r \equiv B_r(x_0)$,

$$\begin{aligned} H(v,\varphi) &= \Delta^{\frac{n}{4}}(v\varphi) - v\Delta^{\frac{n}{4}}\varphi - \varphi\Delta^{\frac{n}{4}} v = \Delta^{\frac{n}{4}}(\eta_{\Lambda r} v\varphi) - \eta_{\Lambda r} v \Delta^{\frac{n}{4}}\varphi - \varphi\Delta^{\frac{n}{4}}\left(\eta_{\Lambda r} v + (1-\eta_{\Lambda r})v\right) \\ &=: I - II - III. \end{aligned}$$

Then we rewrite for a polynomial P of order $\lceil \frac{n}{2} \rceil - 1$ which we will choose below, using again that the support of φ lies in B_r, so $\varphi\eta_{\Lambda r} = \varphi$ on $\mathbb{R}^n$,

$$I = \Delta^{\frac{n}{4}}(\eta_{\Lambda r}(v-P)\varphi) + \Delta^{\frac{n}{4}}(P\varphi),$$
$$II = \eta_{\Lambda r}(v-P)\Delta^{\frac{n}{4}}\varphi + P\Delta^{\frac{n}{4}}\varphi,$$
$$III = \varphi\Delta^{\frac{n}{4}}(\eta_{\Lambda r}(v-P)) + \varphi\Delta^{\frac{n}{4}}(\eta_{\Lambda r} P) + \varphi\Delta^{\frac{n}{4}}((1-\eta_{\Lambda r})v).$$

Thus $I - II - III = \widetilde{I} + \widetilde{II} - \widetilde{III}$, where

$$\begin{aligned} \widetilde{I} &= H(\eta_{\Lambda r}(v-P), \varphi), \\ \widetilde{II} &= \Delta^{\frac{n}{4}}(P\varphi) - P\Delta^{\frac{n}{4}}\varphi, \\ \widetilde{III} &= \varphi\Delta^{\frac{n}{4}}(P + (1-\eta_{\Lambda r})(v-P)). \end{aligned}$$

Theorem 4.2 implies $\|\widetilde{I}\|_{L^2(\mathbb{R}^n)} \prec \|\Delta^{\frac{n}{4}}(\eta_{\Lambda r}(v-P))\|_{L^2}$, Proposition 5.13 states for $u = v$ and $s = \frac{n}{2}$ that

$$\begin{aligned} \|\widetilde{II}\|_{L^2(B_r)} &\prec \|\Delta^{\frac{n}{4}}\eta_{\Lambda r}(v-P)\|_{L^2(\mathbb{R}^n)} + \|\Delta^{\frac{n}{4}} v\|_{L^2(B_{2\Lambda r})} + \Lambda^{-1}\sum_{k=1}^{\infty} 2^{-k} \|\eta^k_{\Lambda r}\Delta^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)} \\ &\prec \|\Delta^{\frac{n}{4}}\eta_{\Lambda r}(v-P)\|_{L^2(\mathbb{R}^n)} + \|\Delta^{\frac{n}{4}} v\|_{L^2(B_{2\Lambda r})} + \Lambda^{-1}\|\Delta^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

It remains to estimate $\widetilde{III}$. Choose P to be the polynomial such that $v - P$ satisfies the mean value condition (3.1) for $N = \lceil \frac{n}{2} \rceil - 1$ and in $B_{2\Lambda r}(x_0)$.
We have to estimate for $\psi \in C_0^\infty(B_r)$, $\|\psi\|_{L^2} \leq 1$,

$$\int \widetilde{III}\psi = \int \psi\varphi \ \Delta^{\frac{n}{4}}(P + (1-\eta_{\Lambda r})(v-P)).$$

Note that
$$P+(1-\eta_{\Lambda r})(v-P)=\eta_{\Lambda r}P+(1-\eta_{\Lambda r}v)\in\mathcal{S}(\mathbb{R}^n),$$
so we can write
$$\int \widetilde{III}\psi \;=\; \int \Delta^{\frac{n}{4}}(\psi\varphi)\ P+(1-\eta_{\Lambda r})(v-P)=\lim_{R\to\infty}\int \Delta^{\frac{n}{4}}(\psi\varphi)\eta_R P+\int \Delta^{\frac{n}{4}}(\psi\varphi)(1-\eta_{\Lambda r})(v-P).$$
By Remark 2.14 we have
$$\int \Delta^{\frac{n}{4}}(\psi\varphi)\eta_R P=o(1)\quad\text{for } R\to\infty,$$
so in fact we only have to estimate for any $R>1$
$$\begin{aligned}\sum_{k=1}^{\infty}\int \psi\ \varphi\ \Delta^{\frac{n}{4}}(\eta_R\eta^k_{\Lambda r}(v-P)) &\overset{L.5.1}{\prec} \sum_{k=1}^{\infty}(2^k\Lambda r)^{-\frac{3}{2}n}\ \|\varphi\|_{L^2}\ \|\eta^k_{\Lambda r}(v-P)\|_{L^1}\\ &\overset{L.2.6}{\prec} \sum_{k=1}^{\infty}(2^k\Lambda)^{-n}r^{-\frac{n}{2}}\ \|\eta^k_{\Lambda r}(v-P)\|_{L^2}\\ &\overset{P.3.11}{\prec} \Lambda^{-\frac{n}{2}}\ \sum_{k=1}^{\infty}2^{-k\frac{n}{2}}(1+k)\ \|\Delta^{\frac{n}{4}}v\|_{L^2(\mathbb{R}^n)}\prec\Lambda^{-\frac{1}{2}}\|\Delta^{\frac{n}{4}}v\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$
In order to finish the whole proof it is then only necessary to apply Lemma 3.5.

Lemma 6.1 □

Lemma 6.2. *For any $v\in H^{\frac{n}{2}}(\mathbb{R}^n)$, $\varepsilon\in(0,1)$, there exists $\Lambda>0$, $R>0$, $\gamma>0$ such that for all $x_0\in\mathbb{R}^n$, $r<R$*
$$\|H(v,v)\|_{L^2(B_r(x_0))}\le\varepsilon\big([v]_{B_{4\Lambda r},\frac{n}{2}}+\|\Delta^{\frac{n}{4}}v\|_{L^2(B_{4\Lambda r})}\big)+C\ \Lambda^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}2^{-\gamma k}\|\Delta^{\frac{n}{4}}v\|_{L^2(A_k)}+\sum_{k=-\infty}^{\infty}2^{-\gamma|k|}[v]_{A_k,\frac{n}{2}}.\right)$$
Here we set $A_k:=B_{2^{k+4}4\Lambda r}\backslash B_{2^{k-1}r}$.

Proof. Let $\delta=\varepsilon\tilde{\delta}>0\in(0,1)$, where $\tilde{\delta}$ is a uniform constant whose value will be chosen later. Pick $\Lambda>10$ depending on δ and v such that
$$\Lambda^{-\frac{1}{2}}\ \|\Delta^{\frac{n}{4}}v\|_{L^2(\mathbb{R}^n)}\le\delta. \tag{6.1}$$
Depending on δ and Λ choose $R>0$ so small such that
$$[v]_{B_{10\Lambda r}(x_0),\frac{n}{2}}+\|\Delta^{\frac{n}{4}}v\|_{L^2(B_{10\Lambda r}(x_0))}\le\delta,\quad\text{for all } x_0\in\mathbb{R}^n,\ r<R. \tag{6.2}$$
We can assume that $v\in C_0^\infty(\mathbb{R}^n)$. In fact, by [Tar07, Lemma 15.10] we can approximate v in $H^{\frac{n}{2}}(\mathbb{R}^n)$ by $v_k\in C_0^\infty(\mathbb{R}^n)$, and one checks that the approximation process does not destroy the argument.
From now on let $r\in(0,R)$ and $x_0\in\mathbb{R}^n$ be arbitrarily fixed and denote $B_r\equiv B_r(x_0)$. Set $P\equiv P_\Lambda\equiv P_{B_{2\Lambda r}}(v)$ the polynomial of degree $N:=\lceil\frac{n}{2}\rceil-1$ such that the mean value condition (3.1) holds on $B_{2\Lambda r}(x_0)$. We denote $\eta_{\Lambda r}\equiv\eta_{\Lambda r,x_0}$ and $\tilde{\eta}_\rho:=\eta_{\rho,0}$.
As P is not a function in $\mathcal{S}(\mathbb{R}^n)$, we "approximate" it by $P^\rho:=\tilde{\eta}_\rho P$, $\rho>\rho_0$ where we choose $\rho_0>2\max\{2\Lambda r+|x_0|,1\}$ such that $B_{\frac{1}{2}\rho_0}(0)\supset\operatorname{supp}v$. Note that in particular, we only work with $\rho>0$ such that
$$\tilde{\eta}_\rho\equiv 1\quad\text{on }\operatorname{supp}\eta_{2\Lambda r,x_0}\cup\operatorname{supp}v,\qquad\text{for all }\rho>\rho_0.$$
Then,
$$v=\tilde{\eta}_\rho v=\eta_{\Lambda r}(v-P)+\tilde{\eta}_\rho(1-\eta_{\Lambda r})(v-P)+P^\rho=:v_\Lambda+v^\rho_{-\Lambda}+P^\rho. \tag{6.3}$$
Observe that all three terms on the right-hand side are functions of $\mathcal{S}(\mathbb{R}^n)$. We have
$$v^2=(v_\Lambda)^2+(v^\rho_{-\Lambda})^2+(P^\rho)^2+2v_\Lambda\ v^\rho_{-\Lambda}+2\left(v_\Lambda+v^\rho_{-\Lambda}\right)\ P^\rho. \tag{6.4}$$
As we want to estimate $H(v,v)$ on $B_r\equiv B_r(x_0)$, we are going to rewrite $H(v,v)\varphi$ for an arbitrary $\varphi\in C_0^\infty(B_r)$, such that $\|\varphi\|_{L^2(\mathbb{R}^n)}\le 1$. For any $\rho>\rho_0$ (with the goal of letting $\rho\to\infty$ in the end), we will use the following facts
$$\varphi P^\rho=\varphi P,\quad v_\Lambda P^\rho=v_\Lambda P,\quad \varphi v^\rho_{-\Lambda}=0.$$

Now we start the rewriting process:

$$\begin{aligned} H(v,v)\varphi &\overset{(6.4)}{=} H(v_\Lambda,v_\Lambda)\varphi \\ &+2\left(\Delta^{\frac{n}{4}}\left(\left(v_\Lambda+v^\rho_{-\Lambda}\right)\ P^\rho\right)-P\ \Delta^{\frac{n}{4}}\left(v_\Lambda+v^\rho_{-\Lambda}\right)\right)\varphi \\ &+\left(\Delta^{\frac{n}{4}}\left(P^\rho\right)^2\right)\varphi \\ &+\left(\Delta^{\frac{n}{4}}(v^\rho_{-\Lambda})^2+2\Delta^{\frac{n}{4}}\left(v_\Lambda\ v^\rho_{-\Lambda}\right)-2v_\Lambda\Delta^{\frac{n}{4}}v^\rho_{-\Lambda}\right)\varphi \\ &-2\left(P\ \Delta^{\frac{n}{4}}P^\rho+v_\Lambda\Delta^{\frac{n}{4}}P^\rho\right)\varphi. \end{aligned}$$

Now we add and substract terms, that vanish for $\rho\to\infty$, and arrive at

$$\begin{aligned} &= H(v_\Lambda,v_\Lambda)\varphi \\ &\quad+2\ \left(\Delta^{\frac{n}{4}}\left(\left(v_\Lambda+v^\rho_{-\Lambda}\right)\ P\right)-P\ \Delta^{\frac{n}{4}}\left(v_\Lambda+v^\rho_{-\Lambda}\right)\right)\varphi \\ &\quad+\left(\Delta^{\frac{n}{4}}\left((\tilde\eta_\rho)^2PP\right)-P\Delta^{\frac{n}{4}}\left((\tilde\eta_\rho)^2P\right)\right)\varphi \\ &\quad+\left(\Delta^{\frac{n}{4}}(v^\rho_{-\Lambda})^2+2\Delta^{\frac{n}{4}}\left(v_\Lambda\ v^\rho_{-\Lambda}\right)-2v_\Lambda\Delta^{\frac{n}{4}}v^\rho_{-\Lambda}\right)\varphi \\ &\quad+\left(P\ \Delta^{\frac{n}{4}}\left((\tilde\eta_\rho)^2P\right)-2\ P\ \Delta^{\frac{n}{4}}P^\rho-2\ v_\Lambda\Delta^{\frac{n}{4}}P^\rho\right)\varphi \\ &\quad+2\ \Delta^{\frac{n}{4}}\left(v^\rho_{-\Lambda}(\tilde\eta_\rho-1)P\right)\ \varphi \\ &=: (I+II+III+IV+V+VI)\,\varphi. \end{aligned}$$

First we treat the terms V and VI which will be the parts vanishing for $\rho\to\infty$. As for V, we have by Remark 2.14,

$$\|\Delta^{\frac{n}{4}}\left((\tilde\eta_\rho)^2P\right)+\|\Delta^{\frac{n}{4}}P^\rho\|_{L^\infty(\mathbb{R}^n)}\le C_{r,\Lambda,v,x_0}\ \rho^{N-\frac{n}{2}}\le C_{r,\Lambda,v,x_0}\rho^{-\frac{1}{2}}.$$

Consequently,

$$\|V\|_{L^2(B_r)}\le C_{r,x_0,v,\Lambda}\rho^{-\frac{1}{2}}.$$

Next, as for VI, the product rule for polynomials, Proposition 5.12 for $M=Id$, $\varphi=v^\rho_{-\Lambda}(\tilde\eta_\rho-1)\in\mathcal{S}(\mathbb{R}^n)$, implies that for some zero-multiplier operator M_β,

$$\Delta^{\frac{n}{4}}\left(v^\rho_{-\Lambda}(\tilde\eta_\rho-1)P\right)=\sum_{|\beta|\le N}\partial^\beta P\ M_\beta\Delta^{\frac{n-2|\beta|}{4}}\left(v^\rho_{-\Lambda}(\tilde\eta_\rho-1)\right).$$

As a consequence, using that P is a polynomial with coefficients depending on Λ,r,v,x_0,

$$\|VI\|_{L^2(B_r)}\le C_{v,r,x_0,\Lambda}\sum_{|\beta|\le N}\|M_\beta\Delta^{\frac{n-2|\beta|}{4}}\left(v^\rho_{-\Lambda}(\tilde\eta_\rho-1)\right)\|_{L^2(B_r)}.$$

Now we use the disjoint support lemma, Lemma 5.1, to estimate for some $k_0=k_0(\rho,x_0,\Lambda)\ge 1$ tending to ∞ as $\rho\to\infty$,

$$\begin{aligned} \|M_\beta\Delta^{\frac{n-2|\beta|}{4}}\left(v^\rho_{-\Lambda}(\tilde\eta_\rho-1)\right)\|_{L^2(B_r)} &\le \sum_{k=k_0}^\infty\|M_\beta\Delta^{\frac{n-2|\beta|}{4}}\left(\eta^k_{\Lambda r,x_0}(v-P)(\tilde\eta_\rho(1-\tilde\eta_\rho))\right)\|_{L^2(B_r)} \\ &\overset{L.5.1}{\le} C_{r,\Lambda}\sum_{k=k_0}^\infty 2^{-k(n-|\beta|)}\|\left(\eta^k_{\Lambda r,x_0}(v-P)\right)\|_{L^2(\mathbb{R}^n)} \\ &\overset{P.3.11}{\le} C_{r,\Lambda}\sum_{k=k_0}^\infty 2^{-k(\frac{n}{2}-N)}(1+|k|)\ \|\Delta^{\frac{n}{4}}v\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

As $N<\frac{n}{2}$, we have proven that

$$\|V\|_{L^2(B_r(x_0))}+\|VI\|_{L^2(B_r(x_0))}=o(1)\quad\text{for }\rho\to\infty.$$

Next, we treat I. By Theorem 4.2 and Lemma 3.5 we have

$$\|I\|_{L^2(B_r)} \prec \|\Delta^{\frac{n}{4}} v_\Lambda\|^2_{L^2(\mathbb{R}^n)} \prec \left([v]_{B_{4\Lambda r},\frac{n}{2}}\right)^2 \overset{(6.2)}{\prec} \delta\ [v]_{B_{4\Lambda r},\frac{n}{2}}.$$

As for II, by Proposition 5.12, for any $w \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}\varphi\big(\Delta^{\frac{n}{4}}(w\ P) - P\Delta^{\frac{n}{4}} w\big) &= \varphi \sum_{1\leq|\beta|\leq N} \partial^\beta P\ M_\beta \Delta^{\frac{n-2|\beta|}{4}} w\\ &\overset{\operatorname{supp}\varphi}{=} \varphi \sum_{1\leq|\beta|\leq N} \Big(\partial^\beta(\eta_{\Lambda r}(P-v))\ M_\beta\Delta^{\frac{n-2|\beta|}{4}} w + \partial^\beta v\ M_\beta \Delta^{\frac{n-2|\beta|}{4}} w\Big),\end{aligned}$$

so

$$\|II\|_{L^2(B_r)} \leq \sum_{1\leq|\beta|\leq N} II^\beta_{1,\Lambda} + II^\beta_{2,\Lambda} + II^\beta_{1,-\Lambda} + II^\beta_{2,-\Lambda},$$

where

$$\begin{aligned}II^\beta_{1,\Lambda} &= \|\partial^\beta(\eta_{\Lambda r}(P-v))\ M_\beta\Delta^{\frac{n-2|\beta|}{4}} v_\Lambda\|_{L^2(B_r)} = \|\partial^\beta v_\Lambda\ M_\beta\Delta^{\frac{n-2|\beta|}{4}} v_\Lambda\|_{L^2(B_r)},\\ II^\beta_{2,\Lambda} &= \|\partial^\beta v\ M_\beta \Delta^{\frac{n-2|\beta|}{4}} v_\Lambda\|_{L^2(B_r)},\\ II^\beta_{1,-\Lambda} &= \|\partial^\beta(\eta_{\Lambda r}(P-v))\ M_\beta\Delta^{\frac{n-2|\beta|}{4}} v^\rho_{-\Lambda}\|_{L^2(B_r)} = \|\partial^\beta v_\Lambda\ M_\beta\Delta^{\frac{n-2|\beta|}{4}} v^\rho_{-\Lambda}\|_{L^2(B_r)},\\ II^\beta_{2,-\Lambda} &= \|\partial^\beta v\ M_\beta\Delta^{\frac{n-2|\beta|}{4}} v^\rho_{-\Lambda}\|_{L^2(B_r)}.\end{aligned}$$

Observe that all the operators involved are of order strictly between $(0,\frac{n}{2})$. Consequently, by Proposition 5.9 and Poincaré's inequality, Lemma 3.5,

$$II^\beta_{1,\Lambda} \prec \left([v]_{B_{4\Lambda r},\frac{n}{2}}\right)^2 \overset{(6.2)}{\prec} \delta\ [v]_{B_{4\Lambda r},\frac{n}{2}}.$$

By Lemma 5.10 and Poincaré's inequality, Lemma 3.5,

$$\begin{aligned}II^\beta_{2,\Lambda} &\prec [v]_{B_{4\Lambda r},\frac{n}{2}} \left(\|\Delta^{\frac{n}{4}} v\|_{L^2(B_{4\Lambda r})} + \Lambda^{-\frac{1}{2}}\|\Delta^{\frac{n}{4}} v\|_{L^2}\right)\\ &\overset{(6.2)\,(6.1)}{\prec} \delta\ \big(\|\Delta^{\frac{n}{4}} v\|_{L^2(B_{4\Lambda r})} + [v]_{B_{4\Lambda r},\frac{n}{2}}\big).\end{aligned}$$

As for $II^\beta_{2,-\Lambda}$ and $II^\beta_{1,-\Lambda}$, we estimate for any $w \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}&\|\partial^\beta w\ M_\beta\Delta^{\frac{n-2|\beta|}{4}} v^\rho_{-\Lambda}\|_{L^2(B_r)}\\ \prec &\sum_{k=1}^\infty \|\partial^\beta \Delta^{-\frac{n}{4}}\big(\eta_{4r}\Delta^{\frac{n}{4}} w\big)\ M_\beta\Delta^{\frac{n-2|\beta|}{4}} \eta^k_{\Lambda r}(v-P)\tilde\eta_\rho\|_{L^2(B_r)}\\ &+ \sum_{l,k=1}^\infty \|\partial^\beta\Delta^{-\frac{n}{4}}\big(\eta^l_{4r}\Delta^{\frac{n}{4}} w\big)\ M_\beta\Delta^{\frac{n-2|\beta|}{4}}\eta^k_{\Lambda r}(v-P)\tilde\eta_\rho\|_{L^2(B_r)} =: \Sigma_1 + \Sigma_2.\end{aligned}$$

We first concentrate on Σ_1. As before, by Lemma 5.1 and using that $1 \leq |\beta| < \frac{n}{2}$,

$$\|\partial^\beta\Delta^{-\frac{n}{4}}\big(\eta_{4r}\Delta^{\frac{n}{4}} w\big)\ M_\beta\Delta^{\frac{n-2|\beta|}{4}}\eta^k_{\Lambda r}(v-P)\tilde\eta_\rho\|_{L^2(B_r)} \overset{L.2.8}{\prec} \Lambda^{|\beta|-\frac{n}{2}}\ \|\eta_{4r}\Delta^{\frac{n}{4}} w\|_{L^2}\ 2^{(|\beta|-n)k}\ (\Lambda r)^{-\frac{n}{2}}\|\eta^k_{\Lambda r}(v-P)\|_{L^2}.$$

Thus, by Proposition 3.11 and as $|\beta| < \frac{n}{2}$ (making $\sum_{k>0} k\ 2^{-k(\frac{n}{2}-|\beta|)}$ convergent),

$$\Sigma_1 \prec \Lambda^{-\frac{1}{2}}\|\Delta^{\frac{n}{4}} w\|_{L^2(B_{4\Lambda r})}\ \|\Delta^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)} \overset{(6.1)}{\prec} \delta\ \|\Delta^{\frac{n}{4}} w\|_{L^2(B_{4\Lambda r})}.$$

For the estimate of Σ_2 we observe

$$\begin{aligned}&\|\partial^\beta\Delta^{-\frac{n}{4}}\big(\eta^l_{4r}\Delta^{\frac{n}{4}} w\big)\ M_\beta\Delta^{\frac{n-2|\beta|}{4}}\eta^k_{\Lambda r}(v-P)\tilde\eta_\rho\|_{L^2(B_r)}\\ \overset{L.5.1}{\prec}\ &(2^l r)^{-\frac{n}{2}-|\beta|}\ \|\big(\eta^l_{4r}\Delta^{\frac{n}{4}} w\big)\|_{L^1}\ \|M_\beta\Delta^{\frac{n-2|\beta|}{4}}\eta^k_{\Lambda r}(v-P)\tilde\eta_\rho\|_{L^2(B_r)}\\ \overset{L.5.1}{\prec}\ &(2^l r)^{-\frac{n}{2}-|\beta|}\ \|\big(\eta^l_{4r}\Delta^{\frac{n}{4}} w\big)\|_{L^1}\ \big(2^k\Lambda r\big)^{-\frac{3}{2}n+|\beta|}\|\eta^k_{\Lambda r}(v-P)\|_{L^1}\ r^{\frac{n}{2}}\\ \prec\ &r^{-\frac{n}{2}}\ 2^{-|\beta| l}\ \|\big(\eta^l_{4r}\Delta^{\frac{n}{4}} w\big)\|_{L^2}\ \big(2^k\Lambda\big)^{-n+|\beta|}\|\eta^k_{\Lambda r}(v-P)\|_{L^2}.\end{aligned}$$

Summing first over k and then over l, using again Proposition 3.11 and that $|\beta| \in [1, N]$,

$$\Sigma_2 \prec \Lambda^{-\frac{n}{2}+N} \sum_{l=1}^{\infty} 2^{-l} \|\eta^l_{4r} \Delta^{\frac{n}{4}} w\|_{L^2}\ \|\Delta^{\frac{n}{4}} v\|_{L^2} \overset{(6.1)}{\prec} \delta \sum_{l=1}^{\infty} 2^{-l} \|\eta^l_{4r} \Delta^{\frac{n}{4}} w\|_{L^2}.$$

So we have shown that

$$\|\partial^\beta w\ M_\beta \Delta^{\frac{n-2|\beta|}{4}} v^\rho_{-\Lambda}\|_{L^2(B_r)} \prec \delta \sum_{l=1}^{\infty} 2^{-l} \|\eta^l_{4r} \Delta^{\frac{n}{4}} w\|_{L^2} + \delta \|\Delta^{\frac{n}{4}} w\|_{L^2(B_{4\Lambda r})} \prec \delta \|\Delta^{\frac{n}{4}} w\|_{L^2(\mathbb{R}^n)}.$$

Setting $w = v$ in the case of $II^\beta_{2,-\Lambda}$ and $w = v_\Lambda$ in the case of $II^\beta_{1,-\Lambda}$, this implies

$$II^\beta_{1,-\Lambda} \prec \delta \|\Delta^{\frac{n}{4}} v_\Lambda\|_{L^2} \prec \delta\ [v]_{B_{4\Lambda r},\frac{n}{2}}, \quad \text{and} \quad II^\beta_{2,-\Lambda} \prec \sum_{l=1}^{\infty} 2^{-l} \|\Delta^{\frac{n}{4}} v\|_{L^2(A_l)} + \delta \|\Delta^{\frac{n}{4}} v\|_{L^2(B_{4\Lambda r})}.$$

<u>As for III</u>, using yet again (6.3), we have $P_\rho \tilde{\eta}_\rho = v - v_\Lambda - v^\rho_{-\Lambda} \tilde{\eta}_\rho$. As a consequence, we can rewrite

$$III = \left(\Delta^{\frac{n}{4}} \left((\tilde{\eta}_\rho)^2 PP\right) - P\Delta^{\frac{n}{4}} \left((\tilde{\eta}_\rho)^2 P\right)\right)\varphi = \left(\Delta^{\frac{n}{4}} \left((v - v_\Lambda - v^\rho_{-\Lambda} \tilde{\eta}_\rho)P\right) - P\Delta^{\frac{n}{4}} \left(v - v_\Lambda - v^\rho_{-\Lambda} \tilde{\eta}_\rho\right)\right)\varphi.$$

Thus, the only part we have not estimated already in II (or which is estimated exactly as in II, as the term containing $v^\rho_{-\Lambda} \tilde{\eta}_\rho$) is $\Delta^{\frac{n}{4}}(vP) - P\Delta^{\frac{n}{4}} v$. Again by Proposition 5.12, this is decomposed into terms of the following form (for $1 \leq |\beta| \leq N$)

$$\begin{aligned}
&\partial^\beta P\ M_\beta \Delta^{\frac{n-2|\beta|}{4}} v \\
=\ & -\partial^\beta((v - P)(1 - \eta_{\Lambda r}))\ M_\beta \Delta^{\frac{n-2|\beta|}{4}} v - \partial^\beta((v - P)\eta_{\Lambda r})\ M_\beta \Delta^{\frac{n-2|\beta|}{4}} v + \partial^\beta v\ M_\beta \Delta^{\frac{n-2|\beta|}{4}} v \\
=:\ & III_1 + III_2 + III_3.
\end{aligned}$$

Of course, $\|III_1\|_{L^2(B_r)} = 0$. By Lemma 5.10,

$$\begin{aligned}
\|III_2\|_{L^2(B_r)} &\prec \|\Delta^{\frac{n}{4}}(v - P)\eta_{\Lambda r}\|_{L^2} \left(\|\Delta^{\frac{n}{4}} v\|_{L^2(B_{2\Lambda r})} + \Lambda^{-\frac{1}{2}} \sum_{k=1}^{\infty} 2^{-\frac{k}{2}} \|\Delta^{\frac{n}{4}} v\|_{L^2(A_k)} \right) \\
&\overset{L.3.5}{\prec} [v]_{\frac{n}{2},4\Lambda r} \left(\|\Delta^{\frac{n}{4}} v\|_{L^2(B_{2\Lambda r})} + \sum_{k=1}^{\infty} 2^{-\frac{k}{2}} \|\Delta^{\frac{n}{4}} v\|_{L^2(A_k)} \right) \\
&\overset{(6.2)}{\prec} \delta [v]_{\frac{n}{2},4\Lambda r} + \delta \sum_{k=1}^{\infty} 2^{-\frac{k}{2}} \|\Delta^{\frac{n}{4}} v\|_{L^2(A_k)}.
\end{aligned}$$

And by Lemma 5.11 and (6.2),

$$\|III_3\|_{L^2(B_r)} \prec \delta \|\Delta^{\frac{n}{4}} v\|_{L^2(B_{4\Lambda r})} + \sum_{k=1}^{\infty} 2^{-\frac{k}{2}} \|\Delta^{\frac{n}{4}} v\|_{L^2(A_k)}.$$

Finally, <u>we have to estimate IV</u>. Set

$$\tilde{A}_k := B_{2^{k+4}\Lambda r} \backslash B_{2^{k-4}\Lambda r}.$$

Using Lemma 5.1 the first term is done as follows (setting P_k to be the polynomial of order N where $v - P_k$ satisfies (3.1) on $B_{2^{k+1}\Lambda r} \backslash B_{2^{k-1}\Lambda r}$)

$$\begin{aligned}
&\|\Delta^{\frac{n}{4}} \left(\eta^k_{\Lambda r}(1 - \eta_{\Lambda r})(\tilde{\eta}_\rho)^2 (v - P)^2\right)\|_{L^2(B_r)} \\
\prec\ & 2^{-k\frac{3}{2}n} \Lambda^{-\frac{3}{2}n} r^{-n} \|\sqrt{\eta^k_{\Lambda r}}(v - P)\|^2_{L^2} \\
\prec\ & 2^{-k\frac{3}{2}n} \Lambda^{-\frac{3}{2}n} r^{-n} \left(\|\sqrt{\eta^k_{\Lambda r}}(v - P_k)\|^2_{L^2} + 2^{nk}(\Lambda r)^n \|\sqrt{\eta^k_{\Lambda r}}(P - P_k)\|^2_{L^\infty} \right) \\
\overset{L.3.6}{\prec}\ & 2^{-k\frac{3}{2}n} \Lambda^{-\frac{3}{2}n} r^{-n} \left((2^k \Lambda r)^n \left([v]_{\tilde{A}_k,\frac{n}{2}}\right)^2 + 2^{nk}(\Lambda r)^n \|\sqrt{\eta^k_{\Lambda r}}(P - P_k)\|^2_{L^\infty} \right) \\
\overset{P.3.10}{\prec}\ & \Lambda^{-\frac{n}{2}}\ 2^{-k\frac{n}{2}} \left(\left([v]_{\tilde{A}_k,\frac{n}{2}}\right)^2 + k \|\sqrt{\eta^k_{\Lambda r}}(P - P_k)\|_{L^\infty}\ \|\Delta^{\frac{n}{4}} v\|_{L^2} \right) \\
\prec\ & \Lambda^{-\frac{n}{2}}\ 2^{-k\frac{n-\frac{1}{4}}{2}} \left(\left([v]_{\tilde{A}_k,\frac{n}{2}}\right)^2 + \|\sqrt{\eta^k_{\Lambda r}}(P - P_k)\|_{L^\infty} \|\Delta^{\frac{n}{4}} v\|_{L^2} \right).
\end{aligned}$$

Note that as $\frac{n}{2} - \frac{1}{8} > \lceil \frac{n}{2} \rceil - 1$, on the one hand Lemma 3.12 is applicable and on the other hand we have by Proposition 2.10

$$\sum_{k=1}^{\infty} 2^{-k\frac{n-\frac{1}{4}}{2}} \left([v]_{\tilde{A}_k,\frac{n}{2}}\right)^2 \prec \|\Delta^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)} \sum_{k=1}^{\infty} 2^{-k\frac{n-\frac{1}{4}}{2}} [v]_{\tilde{A}_k,\frac{n}{2}}.$$

Consequently, we have for some $\gamma > 0$

$$\|\Delta^{\frac{n}{4}}(v^{\rho}_{-\Lambda})^2\|_{L^2(B_r)} \prec \left(1 + \|\Delta^{\frac{n}{4}} v\|_{L^2}\right) \sum_{k=-\infty}^{\infty} 2^{-\gamma|k|}[v]_{\tilde{A}_k,\frac{n}{2}} \overset{(6.1)}{\prec} \Lambda^{\frac{1}{2}} \sum_{k=-\infty}^{\infty} 2^{-\gamma|k|}[v]_{\tilde{A}_k,\frac{n}{2}}.$$

For the next term in IV, using the disjoint support as well as Poincaré's inequality, Lemma 2.6 and Lemma 3.5, and the estimate on mean value polynomials, Proposition 3.11, and as

$$v_\Lambda v^{\rho}_{-\Lambda} = \sum_{k=1}^{3} v_\Lambda \left(\eta^k_{\Lambda r} \tilde{\eta}_\rho \ (v - P)\right),$$

we can estimate

$$\begin{aligned}
\|\Delta^{\frac{n}{4}}\left(v_\Lambda\ v^{\rho}_{-\Lambda}\right)\|_{L^2(B_r)} &\overset{L.5.1}{\leq} \sum_{k=1}^{3} \left(2^k \Lambda r\right)^{-\frac{3}{2}n} \|v_\Lambda\|_{L^2}\ \|\eta^k_{\Lambda r}(v-P)\|_{L^2}\ r^{\frac{n}{2}}\\
&\overset{L.2.6}{\prec} \sum_{k=1}^{3} \left(2^k \Lambda r\right)^{-\frac{3}{2}n} \ (\Lambda r)^{\frac{n}{2}} \|\Delta^{\frac{n}{4}} v_\Lambda\|_{L^2}\ \|\eta^k_{\Lambda r}(v-P)\|_{L^2}\ r^{\frac{n}{2}}\\
&\overset{L.3.5,\ P.3.11}{\prec} \Lambda^{-\frac{n}{2}} [v]_{B_{4\Lambda r},\frac{n}{2}}\ \|\Delta^{\frac{n}{4}} v\|_{L^2(\mathbb{R}^n)} \overset{(6.1)}{\prec} \delta\ [v]_{B_{4\Lambda r},\frac{n}{2}}.
\end{aligned}$$

Last but not least,

$$\begin{aligned}
\|v_\Lambda \Delta^{\frac{n}{4}} \eta^k_{\Lambda r}(v-P)\tilde{\eta}_\rho\|_{L^2(B_r)} &\overset{L.5.1}{\prec} (2^k \Lambda r)^{-n} \|v_\Lambda\|_{L^2}\ \|\eta^k_{\Lambda r}(v-P)\|_{L^2}\\
&\overset{L.2.6,\ L.3.5}{\prec} 2^{-nk} (\Lambda r)^{-\frac{n}{2}} [v]_{B_{4\Lambda r},\frac{n}{2}}\ \|\eta^k_{\Lambda r}(v-P)\|_{L^2}\\
&\overset{(6.2)}{\prec} 2^{-k\frac{n}{2}} \delta \Big(\left(2^k \Lambda r\right)^{-\frac{n}{2}} \|\eta^k_{\Lambda r}(v-P_k)\|_{L^2} + \|\eta^k_{\Lambda r}(P-P_k)\|_{L^\infty}\Big)\\
&\overset{L.3.6}{\prec} \delta\left(2^{-\frac{n}{2}k}\ [v]_{A_k,\frac{n}{2}} + 2^{-\frac{n}{2}k} \|\eta^k_{\Lambda r}(P-P_k)\|_{L^\infty}\right).
\end{aligned}$$

Again, as $\frac{n}{2} > N$, Lemma 3.12 implies that for some $\gamma > 0$.

$$\|v_\Lambda \Delta^{\frac{n}{4}} v_{-\Lambda}\|_{L^2(B_r)} \prec \sum_{k=-\infty}^{\infty} 2^{-\gamma|k|}[v]_{A_k,\frac{n}{2}}.$$

We conclude by taking $\delta = \tilde{\delta}\varepsilon$ for a uniformly small $\tilde{\delta} > 0$ which does *not* depend on Λ or $\|\Delta^{\frac{n}{4}} v\|_{L^2}$. □

7 Euler-Lagrange Equations

As in [DLR09] we will have two equations controlling the behavior of a critical point of E_n. First of all, we are going to use a different structure equation: Obviously, for any $u \in H^{\frac{n}{2}}(\mathbb{R}^n, \mathbb{R}^m)$ with $u(x) \in \mathbb{S}^{m-1}$ almost everywhere on a domain $D \subset \mathbb{R}^n$, we have for $w := \eta u$, $\eta \in C_0^\infty(D)$,

$$w \cdot \Delta^{\frac{n}{4}} w = -\frac{1}{2} H(w, w) + \frac{1}{2} \Delta^{\frac{n}{4}} \eta^2. \qquad (7.1)$$

The Euler-Lagrange Equations are computed similar as in [DLR09], [Hél02]. As we want to localize them, we apply also Lemma 5.5.

Proposition 7.1 (Localized Euler-Lagrange Equation)**.** *Let $\eta \in C_0^\infty(D)$ and $\eta \equiv 1$ in a neighborhood of some ball $\tilde{D} \subset D$.*

Let $u \in H^{\frac{n}{2}}(\mathbb{R}^n, \mathbb{R}^m)$ be a critical point of $E_n(\cdot)$ on D, cf. Definition 1.1. Then $w := \eta u$ satisfies for every $\psi_{ij} \in C_0^\infty(\tilde{D})$, such that $\psi_{ij} = -\psi_{ji}$,

$$-\int\limits_{\mathbb{R}^n} w^i \ \Delta^{\frac{n}{4}} w^j \ \Delta^{\frac{n}{4}} \psi_{ij} = -\int\limits_{\mathbb{R}^n} a_{ij}\psi_{ij} + \int\limits_{\mathbb{R}^n} \Delta^{\frac{n}{4}} w^j \ H(w^i, \psi_{ij}). \tag{7.2}$$

Here $a \in L^2(\mathbb{R}^n)$ depends on the choice of η.

Note that this result holds also if $u \in L^\infty(\mathbb{R}^n)$ and $\Delta^{\frac{n}{4}} u \in L^2(\mathbb{R}^n)$, the setting of [DLR09], by adapting the proof of Lemma 5.5.

Proof of Proposition 7.1.

By the standard argument (cf. [DLR09]), for any $v \in H^{\frac{n}{2}}(\mathbb{R}^n, \mathbb{R}^m)$ such that $\operatorname{supp} v \subset \overline{D}$ and $v \in T_u\mathbb{S}^{m-1}$ a.e.

$$\int\limits_{\mathbb{R}^n} \Delta^{\frac{n}{4}} u \cdot \Delta^{\frac{n}{4}} v = 0. \tag{7.3}$$

Let $\psi_{ij} \in C_0^\infty(\tilde{D}, \mathbb{R})$, $1 \leq i,j \leq m$, $\psi_{ij} = -\psi_{ij}$. Then $v^j := \psi_{ij} u^i \in H^{\frac{n}{2}}(\mathbb{R}^n)$, $1 \leq j \leq m$. Moreover, $u \cdot v = 0$. As for $x \in D$ the vector $u(x) \in \mathbb{R}^m$ is orthogonal to the tangential space of $\mathbb{S}^{m-1}$ at the point $u(x)$, this implies $v \in T_u\mathbb{S}^{m-1}$. Consequently, (7.3) holds for this specific v. Let η be the cutoff function from above, i.e. $\eta \in C_0^\infty(D)$, $\eta \equiv 1$ on an open neighborhood of the ball $\tilde{D} \subset D$ and set $w := \eta u$. Because of $\operatorname{supp} \psi \subset \tilde{D}$ we have that $v^j = w^i \psi_{ij}$. Thus,

$$\int\limits_{\mathbb{R}^n} \Delta^{\frac{n}{4}} w^j \ \Delta^{\frac{n}{4}} (w^i \psi_{ij}) \overset{(7.3)}{=} \int\limits_{\mathbb{R}^n} \Delta^{\frac{n}{4}} (w^j - u^j) \ \Delta^{\frac{n}{4}} (w^i \psi_{ij}). \tag{7.4}$$

Observe that $w^i \in L^\infty(\mathbb{R}^n) \cap H^{\frac{n}{2}}(\mathbb{R}^n)$ and by choice of η and $\tilde{D}$, the distance $\operatorname{dist}\left(\operatorname{supp}(w^j - u^j), \tilde{D}\right) > 0$. Hence, Lemma 5.5 implies that there is $a_{ij} := \tilde{a}_j w^i \in L^2(\mathbb{R}^n)$ such that

$$\int\limits_{\mathbb{R}^n} \Delta^{\frac{n}{4}} (w^j - u^j) \ \Delta^{\frac{n}{4}} (w^i \varphi) = \int\limits_{\mathbb{R}^n} a_{ij}\varphi \quad \text{for all } \varphi \in C_0^\infty(\tilde{D}). \tag{7.5}$$

As a consequence, (7.4) can be written as

$$\int\limits_{\mathbb{R}^n} \Delta^{\frac{n}{4}} w^j \ \Delta^{\frac{n}{4}} (w^i \psi_{ij}) = \int\limits_{\mathbb{R}^n} a_{ij}\psi_{ij}, \quad \text{for every } \psi_{ij} \in C_0^\infty(\tilde{D}) \text{ such that } \psi_{ij} = -\psi_{ji}. \tag{7.6}$$

Moving on, we have just by the definition of $H(\cdot,\cdot)$,

$$\Delta^{\frac{n}{4}} (w^i \psi_{ij}) = \Delta^{\frac{n}{4}} w^i \ \psi_{ij} + w^i \ \Delta^{\frac{n}{4}} \psi_{ij} + H(w^i, \psi_{ij}). \tag{7.7}$$

Hence, putting (7.6) and (7.7) together

$$\begin{aligned} -\int\limits_{\mathbb{R}^n} w^i \ \Delta^{\frac{n}{4}} w^j \ \Delta^{\frac{n}{4}} \psi_{ij} \quad &= \quad -\int\limits_{\mathbb{R}^n} a_{ij}\psi_{ij} + \int\limits_{\mathbb{R}^n} \Delta^{\frac{n}{4}} w^j \ \Delta^{\frac{n}{4}} w^i \ \psi_{ij} + \int\limits_{\mathbb{R}^n} \Delta^{\frac{n}{4}} w^j \ H(w^i, \psi_{ij}) \\ &\overset{\psi_{ij} = -\psi_{ji}}{=} \quad -\int\limits_{\mathbb{R}^n} a_{ij}\psi_{ij} + \int\limits_{\mathbb{R}^n} \Delta^{\frac{n}{4}} w^j \ H(w^i, \psi_{ij}). \end{aligned}$$

Proposition 7.1 □

8 Homogeneous Norm for the Fractional Sobolev Space

Recall from Section 2.3 the definition of the "homogeneous norm" $[u]_{D,s}$. The goal of this section is the following lemma which compares for balls B the size of $[u]_{B,\frac{n}{2}}$ to the size of $\|\Delta^{\frac{n}{4}} u\|_{L^2(B)}$. Obviously, these two semi-norms are not equivalent. In fact, take for instance any nonzero $u \in H^{\frac{n}{2}}(\mathbb{R}^n)$ with support outside of B. Then $[u]_{B,\frac{n}{2}}$ vanishes, but $\Delta^{\frac{n}{4}} u$ can not be constantly zero (cf. Lemma 2.4). Anyway, these two semi-norms can be compared in the following sense:

Lemma 8.1. *There is a uniform $\gamma > 0$ such that for any $\varepsilon > 0$, $n \in \mathbb{N}$, there exists a constant $C_\varepsilon > 0$ such that for any $v \in \mathcal{S}(\mathbb{R}^n)$, $B_r \equiv B_r(x) \subset \mathbb{R}^n$*

$$[v]_{B_r,\frac{n}{2}} \leq \varepsilon [v]_{B_{8r},\frac{n}{2}} + C_\varepsilon \Big[\Vert \Delta^{\frac{n}{4}} v \Vert_{L^2(B_{16r})} + \sum_{k=1}^{\infty} 2^{-nk} \Vert \eta_{8r}^k \Delta^{\frac{n}{4}} v \Vert_{L^2} + \sum_{j=-\infty}^{\infty} 2^{-\gamma |j|}\ [v]_{\tilde{A}_j,\frac{n}{2}} \Big]$$

where $\tilde{A}_j = B_{2^{j+5}r} \backslash B_{2^{j-5}r}$.

Proof of Lemma 8.1.
Set $N := \lceil \frac{n}{2} \rceil - 1$, $s := \frac{n}{2} - N \in \{\frac{1}{2}, 1\}$, and let P_{2r} be the polynomial of degree N such that the mean value condition (3.1) holds for N and B_{2r}. Let at first n be odd. Set $\tilde{v} := \eta_{2r}(v - P_{2r})$. Note that

$$\tilde{v} = v - P_{2r} \quad \text{on } B_r. \tag{8.1}$$

Consequently,

$$\begin{aligned} \left([v]_{B_r,\frac{n}{2}}\right)^2 &\overset{(8.1)}{=} \left([\tilde{v}]_{B_r,\frac{n}{2}}\right)^2 \overset{s:=\frac{1}{2}}{\leq} \sum_{|\alpha|=N} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \frac{(\partial^\alpha \tilde{v}(x) - \partial^\alpha \tilde{v}(y))(\partial^\alpha \tilde{v}(x) - \partial^\alpha \tilde{v}(y))}{|x-y|^{n+2s}}\, dx\, dy \\ &\overset{P.2.10}{\approx} \sum_{|\alpha|=N} \int\limits_{\mathbb{R}^n} \Delta^{\frac{s}{2}} \partial^\alpha \tilde{v}\ \Delta^{\frac{s}{2}} \partial^\alpha \tilde{v}. \end{aligned}$$

Thus,

$$\left([v]_{B_r,\frac{n}{2}}\right)^2 \prec \Vert \Delta^{\frac{n}{4}} \tilde{v} \Vert_{L^2} \sup_{\substack{\varphi \in C_0^\infty(B_{4r}(0)) \\ \Vert \Delta^{\frac{n}{4}} \varphi \Vert_{L^2} \leq 1}} \int\limits_{\mathbb{R}^n} \Delta^{\frac{n}{4}} \tilde{v}\ M \Delta^{\frac{n}{4}} \varphi,$$

where M is a zero-multiplier operator. By a similar argument this also holds for n even. Using Young's inequality,

$$[v]_{B_r,\frac{n}{2}} \prec \varepsilon \Vert \Delta^{\frac{n}{4}} \tilde{v} \Vert_{L^2} + \frac{1}{\varepsilon} \sup_{\substack{\varphi \in C_0^\infty(B_{4r}) \\ \Vert \Delta^{\frac{n}{4}} \varphi \Vert_{L^2} \leq 1}} \int\limits_{\mathbb{R}^n} \Delta^{\frac{n}{4}} \tilde{v}\ M \Delta^{\frac{n}{4}} \varphi \overset{L.3.5}{\prec} \varepsilon [v]_{B_{8r},\frac{n}{2}} + \frac{1}{\varepsilon} \sup_{\substack{\varphi \in C_0^\infty(B_{4r}) \\ \Vert \Delta^{\frac{n}{4}} \varphi \Vert_{L^2} \leq 1}} \int\limits_{\mathbb{R}^n} \Delta^{\frac{n}{4}} \tilde{v}\ M \Delta^{\frac{n}{4}} \varphi.$$

For such a $\varphi \in C_0^\infty(B_{4r})$, $\Vert \Delta^{\frac{n}{4}} \varphi \Vert_{L^2} \leq 1$ we decompose

$$\begin{aligned} \int\limits_{\mathbb{R}^n} \Delta^{\frac{n}{4}} \tilde{v}\ M \Delta^{\frac{n}{4}} \varphi &\overset{P.2.15}{=} \int\limits_{\mathbb{R}^n} \Delta^{\frac{n}{4}} v\ \eta_{8r} M \Delta^{\frac{n}{4}} \varphi + \sum_{k=1}^{\infty} \int\limits_{\mathbb{R}^n} \Delta^{\frac{n}{4}} v\ \eta_{8r}^k M \Delta^{\frac{n}{4}} \varphi - \sum_{k=1}^{\infty} \int\limits_{\mathbb{R}^n} \Delta^{\frac{n}{4}} \left(\eta_{2r}^k (v - P_{2r})\right)\ M \Delta^{\frac{n}{4}} \varphi \\ &=: I + \sum_{k=1}^{\infty} II_k - \sum_{k=1}^{\infty} III_k. \end{aligned}$$

In fact, to apply Proposition 2.15 or Remark 2.14 correctly, we should have used a similar argument as in the proof of Lemma 6.2. Obviously, using Hörmander's theorem [Hör60],

$$|I| \prec \Vert \Delta^{\frac{n}{4}} v \Vert_{L^2(B_{8r})}.$$

Moreover, for any $k \in \mathbb{N}$ by Lemma 5.1 and Poincaré's inequality, Lemma 2.6,

$$|II_k| \prec \left(2^k r\right)^{-n}\ \Vert \eta_{8r}^k \Delta^{\frac{n}{4}} v \Vert_{L^2}\ r^n = 2^{-nk}\ \Vert \eta_{8r}^k \Delta^{\frac{n}{4}} v \Vert_{L^2}.$$

As for III_k, let for $k \in \mathbb{N}$, P_{2r}^k the polynomial which makes $v - P_{2r}^k$ satisfy the mean value condition (3.1) on $B_{2^{k+2}r} \backslash B_{2^k r}$. If $k \geq 3$,

$$\begin{aligned} |III_k| &\overset{L.5.1}{\prec} r^{-\frac{n}{2}}\ \left(2^k\right)^{-\frac{3}{2}n}\ \Vert \eta_{2r}^k (v - P_{2r}) \Vert_{L^2} \prec r^{-\frac{n}{2}}\ 2^{-\frac{3}{2}nk}\ \left(\Vert \eta_{2r}^k (v - P_{2r}^k) \Vert_{L^2} + 2^{k\frac{n}{2}} r^{\frac{n}{2}} \Vert \eta_{2r}^k (P_{2r} - P_{2r}^k) \Vert_{L^\infty}\right) \\ &\overset{L.3.6}{\prec} 2^{-nk}\ \left([v]_{\tilde{A}_k,\frac{n}{2}} + \Vert \eta_{2r}^k (P_{2r} - P_{2r}^k) \Vert_{L^\infty}\right). \end{aligned}$$

This and Lemma 3.12 imply for a $\gamma > 0$, $\sum_{k=3}^{\infty} III_k \prec \sum\limits_{j=-\infty}^{\infty} 2^{-|j|\gamma}\ [v]_{\tilde{A}_j,\frac{n}{2}}$. It remains to estimate III_1, III_2 (where we can not use the disjoint support lemma, Lemma 5.1). Let from now on $k = 1$ or $k = 2$. By Lemma 3.6

$$\begin{aligned} III_k &\leq \Vert \Delta^{\frac{n}{4}} \left(\eta_{2r}^k (v - P_{2r}^k)\right) \Vert_{L^2} + \Vert \Delta^{\frac{n}{4}} \left(\eta_{2r}^k \left(P_{2r}^k - P_{2r}\right)\right) \Vert_{L^2} \\ &\prec [v]_{\tilde{A}_k,\frac{n}{2}} + \Vert \Delta^{\frac{n}{4}} \left(\eta_{2r}^k \left(P_{2r}^k - P_{2r}\right)\right) \Vert_{L^2}. \end{aligned}$$

The following will be similar to the calculations in the proof of Lemma 3.5 and Proposition 3.4. Set

$$w^k_{\alpha,\beta} := \partial^\alpha \eta^k_{2r}\, \partial^\beta \big(P^k_{2r} - P_{2r}\big).$$

We calculate for odd $n \in \mathbb{N}$,

$$\|\Delta^{\frac{n}{4}} \big(\eta^k_{2r}\big(P^k_{2r} - P_{2r}\big)\big)\|^2_{L^2} \prec \sum_{|\alpha|+|\beta|=\frac{n-1}{2}} [w^k_{\alpha,\beta}]^2_{\mathbb{R}^n,\frac{1}{2}}.$$

As $\operatorname{supp} w^k_{\alpha,\beta} \subset B_{2^{k+2}r}\backslash B_{2^k r}$, so one can check that $[w^k_{\alpha,\beta}]^2_{\mathbb{R}^n,\frac{1}{2}} \prec \max_{|\delta|\leq \frac{n+1}{2}} r^{2|\delta|} \|\partial^\delta(P_{2r} - P^k_{2r})\|^2_{L^\infty(\operatorname{supp}\eta^k_{2r})}$. Taking the square root, we have shown that

$$\sum_{k=1}^{2} \|\Delta^{\frac{n}{4}} \big(\eta^k_{2r}\big(P^k_{2r} - P_{2r}\big)\big)\|_{L^2} \prec \max_{|\delta|\leq N} r^{|\delta|} \sum_{k=1}^{2} \|\partial^\beta(P_{2r} - P^k_{2r})\|_{L^\infty(\operatorname{supp}\eta^k_{2r})}.$$

Of course, the same holds true if $n \in \mathbb{N}$ is even. Now, in the proof of Lemma 3.12, more precisely in (3.6), it was shown that

$$\begin{aligned} \sum_{k=1}^{2} \|\partial^\delta(P_{2r} - P^k_{2r})\|_{L^\infty(\tilde{A}_k)} &\prec \sum_{k=1}^{\infty} 2^{-nk}\ \|\partial^\delta(P_{2r} - P^k_{2r})\|_{L^\infty(\tilde{A}_k)} \\ = \sum_{k=1}^{\infty} 2^{-nk}\ \|\partial^\delta(Q^{|\delta|}_{2r} - Q^{|\delta|}_{k})\|_{L^\infty(\tilde{A}_k)} &\overset{(3.6)}{\prec} r^{-|\delta|} \sum_{j=-\infty}^{\infty} 2^{-\gamma|j|}[v]_{\tilde{A}_j,\frac{n}{2}}. \end{aligned}$$

This concludes the proof.

Lemma 8.1 □

Moreover, the following decomposing result holds:

Lemma 8.2. *([DLR09, Theorem A.1])*
For any $s > 0$ there is a constant $C_s > 0$ such that the following holds. For any $v \in \mathcal{S}(\mathbb{R}^n)$, $r > 0$, $x \in \mathbb{R}^n$,

$$\big([v]_{B_r(x),s}\big)^2 \leq C_s \sum_{k=-\infty}^{-1} \big([v]_{A_k,s}\big)^2.$$

Here A_k denotes $B_{2^{k+1}r}(x)\backslash B_{2^{k-1}r}(x)$.

Remark 8.3. *By the same reasoning as in Lemma 8.2, one can also see that for two Annuli-families of different width, say $A_k := B_{2^{k+\lambda}r}\backslash B_{2^{k-\lambda}r}$ and $\tilde{A}_k := B_{2^{k+\Lambda}r}\backslash B_{2^{k-\Lambda}r}$ we can compare $[v]_{A_k,s} \leq C_{\lambda,\Lambda,s} \sum_{l=k-N_{\lambda,\Lambda}}^{k+N_{\lambda,\Lambda}} [v]_{\tilde{A}_l,s}$. In particular we don't have to be too careful about the actual choice of the width of the family A_k for quantities like $\sum_{k=-\infty}^{\infty} 2^{-\gamma|k|}[v]_{A_k,s}$, as long as we can afford to deal with constants depending on the change of width, i.e. if we can afford to have e.g. $C_{\Lambda,\lambda,\gamma,s} \sum_{l=-\infty}^{\infty} 2^{-\gamma|l|}[v]_{\tilde{A}_l,s}$.*

9 Growth Estimates: Proof of Theorem 1.2

In this section, we derive estimates from equations (7.1) and (7.2), similar to the usual Dirichlet-Growth estimates.

Lemma 9.1. *Let $w \in H^{\frac{n}{2}}(\mathbb{R}^n,\mathbb{R}^m)$, $\varepsilon > 0$. Then there exist constants $\Lambda > 0$, $R > 0$, $\gamma > 0$ such that if w is a solution of (7.1), then for any $x \in \mathbb{R}^n$, $r \in (0,R)$, $A_k = B_{2^{k+1}r}(x_0)\backslash B_{2^{k-1}r}(x_0)$,*

$$\begin{aligned} \|w \cdot \Delta^{\frac{n}{4}} w\|_{L^2(B_r(x_0))} &\leq \varepsilon\big(\|\Delta^{\frac{n}{4}} w\|_{L^2(B_{4\Lambda r})} + [w]_{B_{4\Lambda r},\frac{n}{2}}\big) \\ &\quad + C_{\Lambda,w}\left(r^{\frac{n}{2}} + \sum_{k=1}^{\infty} 2^{-\gamma k} \|\Delta^{\frac{n}{4}} w\|_{L^2(A_k)} + \sum_{k=-\infty}^{\infty} 2^{-\gamma|k|}[w]_{A_k,\frac{n}{2}} \right). \end{aligned}$$

***Proof of Lemma* 9.1.**
As $\Delta^{\frac{n}{4}}\eta^2$ is bounded (cf. the proof of Proposition 2.13), $\|\Delta^{\frac{n}{4}}\eta^2\|_{L^2(B_r)} \leq C_\eta r^{\frac{n}{2}}$. The result follows by applying Lemma 6.2 in (7.1), using also Remark 8.3.

Lemma 9.1 □

The next Lemma is a simple consequence of Hölder and Poincaré inequality, Lemma 2.6.

Lemma 9.2. *Let $a \in L^2(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} a\ \varphi \leq C\ r^{\frac{n}{2}}\ \|a\|_{L^2(\mathbb{R}^n)}\ \|\Delta^{\frac{n}{4}}\varphi\|_{L^2(\mathbb{R}^n)}$$

for any $\varphi \in C_0^\infty(B_r(x_0))$, $r > 0$.

Lemma 9.3. *For any $w \in H^{\frac{n}{2}} \cap L^\infty(\mathbb{R}^n, \mathbb{R}^m)$ and any $\varepsilon > 0$ there are constants $\Lambda > 0$, $R > 0$ such that if w is a solution to (7.2) for some smoothly bounded domain $\tilde{D} \subset \mathbb{R}^n$ then for any $B_{\Lambda r}(x) \subset \tilde{D}$, $r \in (0, R)$ and any skew-symmetric $\alpha \in \mathbb{R}^{n\times n}$, $|\alpha| \leq 2$,*

$$\|w^i\alpha_{ij}\Delta^{\frac{n}{4}}w^j\|_{L^2(B_r)} \leq \varepsilon\|\Delta^{\frac{n}{4}}w\|_{B_{\Lambda r}(x)} + C_{\varepsilon,\tilde{D},w}\left(r^{\frac{n}{2}} + \sum_{k=1}^{\infty} 2^{-nk}\ \|\Delta^{\frac{n}{4}}w\|_{L^2(A_k)}\right).$$

Here, $A_k = B_{2^{k+1}r}(x_0)\backslash B_{2^{k-1}r}(x_0)$.

***Proof of Lemma* 9.3.**
Let $\delta = C\varepsilon > 0$ for a uniform constant C which will be clear later. Set $\Lambda_1 > 1$ ten times the uniform constant Λ from Theorem 1.6 and choose $\Lambda_2 > 10$, $\Lambda := 10\Lambda_1\Lambda_2$, $R > 0$ such that

$$(\Lambda_2)^{-\frac{1}{2}}\|\Delta^{\frac{n}{4}}w\|_{L^2(\mathbb{R}^n)} \leq \delta, \tag{9.1}$$

$$[w]_{B_{10\Lambda r},\frac{n}{2}} + \|\Delta^{\frac{n}{4}}w\|_{L^2(B_{10\Lambda r})} \leq \delta \quad \text{for any } x \in \mathbb{R}^n,\ r \in (0, R). \tag{9.2}$$

Fix now any $r \in (0, R)$, $x \in \mathbb{R}^n$ such that $B_{\Lambda r}(x) \subset \tilde{D}$. Set $v := w^i\alpha_{ij}\Delta^{\frac{n}{4}}w^j$. By Theorem 1.6

$$\|\eta_r v\|_{L^2} \leq C \sup_{\substack{\varphi\in C_0^\infty(B_{\Lambda_1 r}(x))\\ \|\Delta^{\frac{n}{4}}\varphi\|_{L^2}\leq 1}} \int \eta_r\ v\ \Delta^{\frac{n}{4}}\varphi.$$

We have for such a $\varphi \in C_0^\infty(B_{\Lambda_1 r}(x))$, $\|\Delta^{\frac{n}{4}}\varphi\|_{L^2} \leq 1$,

$$\int_{\mathbb{R}^n} \eta_r v\ \Delta^{\frac{n}{4}}\varphi = \int v\ \Delta^{\frac{n}{4}}\varphi + \int (\eta_r - 1)\ v\ \Delta^{\frac{n}{4}}\varphi =: I + II.$$

In order to estimate II, we use the compact support of φ in $B_{\Lambda_1 r}$ and apply Corollary 5.2 and Poincaré's inequality, Lemma 2.6. First for all big $k \geq K_{\Lambda_1}$, then also for any other $k \in \mathbb{N}$ we have

$$II = \int (\eta_r - 1)v\ \Delta^{\frac{n}{4}}\varphi \overset{\substack{C.5.2\\ L.2.6}}{\leq} C_{\Lambda_1}\sum_{k=1}^{\infty} 2^{-nk}\ \|\eta_r^k v\|_{L^2}\ \|\Delta^{\frac{n}{4}}\varphi\|_{L^2(\mathbb{R}^n)} \leq C_{\Lambda_1}\|w\|_{L^\infty}\sum_{k=1}^{\infty} 2^{-nk}\ \|\eta_r^k\Delta^{\frac{n}{4}}w\|_{L^2}.$$

The remaining term I is controlled by the PDE (7.2), setting $\psi_{ij} := \alpha_{ij}\varphi$ which is an admissible test function:

$$\begin{aligned} I &\overset{(7.2)}{=} \int_{\mathbb{R}^n} a_{ij}\ \alpha_{ij}\ \varphi + \alpha_{ij}\int_{\mathbb{R}^n} \Delta^{\frac{n}{4}}w^j\ H(w^i, \varphi) \\ &=: \quad I_1 + \alpha_{ij}\int_{\mathbb{R}^n} \eta_{4\Lambda_1 r}\ \Delta^{\frac{n}{4}}w^j\ H(w^i, \varphi) + \alpha_{ij}\sum_{k=1}^{\infty}\int_{\mathbb{R}^n} \eta_{4\Lambda_1 r}^k\ \Delta^{\frac{n}{4}}w^j\ H(w^i, \varphi) =: I_1 + I_2 + \sum_{k=1}^{\infty} I_{3,k}. \end{aligned}$$

By Lemma 9.2, $I_1 \leq C_{\Lambda_1} r^{\frac{n}{2}}\ \|a\|_{L^2}$. By Lemma 6.1 (taking $r = \Lambda_1 r$ and $\Lambda = \Lambda_2$) and the choice of Λ_2 and R, (9.1) and (9.2), $I_2 \prec \delta\ \|\eta_{4\Lambda_2 r}\Delta^{\frac{n}{4}}w\|_{L^2}$. As for $I_{3,k}$, because the support of φ and $\eta_{4\Lambda_1 r}^k$ is disjoint, by Lemma 5.1,

$$\int_{\mathbb{R}^n} \eta_{4\Lambda_1 r}^k\Delta^{\frac{n}{4}}w^j H(w^i, \varphi) = \int_{\mathbb{R}^n} \eta_{4\Lambda_1 r}^k\Delta^{\frac{n}{4}}w^j\left(\Delta^{\frac{n}{4}}(w^i\varphi) - w^i\Delta^{\frac{n}{4}}\varphi\right) \overset{L.5.1}{\prec} C_{\Lambda_1}\ \|w\|_{L^\infty}\ 2^{-nk}\ \|\eta_{4\Lambda_1 r}^k\Delta^{\frac{n}{4}}w^j\|_{L^2}.$$

Using Remark 8.3 we conclude.

Lemma 9.3 □

Lemma 9.4. *Let $w \in H^{\frac{n}{2}} \cap L^\infty(\mathbb{R}^n, \mathbb{R}^m)$ satisfy (7.1) and (7.2) (for some smoothly bounded domain $\tilde{D}$, and some η). Assume furthermore that $w(y) \in \mathbb{S}^{m-1}$ for almost every $y \in \tilde{D}$. Then for any $\varepsilon > 0$ there is $\Lambda > 0$, $R > 0$ and $\gamma > 0$, such that for all $r \in (0,R)$, $x \in \mathbb{R}^n$ such that $B_{\Lambda r}(x) \subset \tilde{D}$,*

$$[w]_{B_r,\frac{n}{2}} + \|\Delta^{\frac{n}{4}} w\|_{L^2(B_r)} \leq \varepsilon\left([w]_{B_{\Lambda r},\frac{n}{2}} + \|v\|_{L^2(B_{\Lambda r})}\right) + C_\varepsilon \left(\sum_{k=-\infty}^{\infty} 2^{-\gamma|k|} \left([w]_{A_k,\frac{n}{2}} + \|\Delta^{\frac{n}{4}} w\|_{L^2(A_k)}\right) r^{\frac{n}{2}} \right).$$

Here, $A_k = B_{2^{k+1}r}(x_0) \backslash B_{2^{k-1}r}(x_0)$.

Proof of Lemma 9.4.
Let $\varepsilon > 0$ be given and $\delta := \delta_\varepsilon$ to be chosen later. Take from Lemma 9.1 and Lemma 9.3 the smallest R to be our $R > 0$ and the biggest Λ to be our $\Lambda > 10$, such that the following holds: For any skew symmetric matrix $\alpha \in \mathbb{R}^{n\times n}$, $|\alpha| \leq 2$ and any $B_{\Lambda r}(x) \equiv B_\Lambda \subset \tilde{D}$, $r \in (0,R)$ and for a certain $\gamma > 0$

$$\begin{aligned} & \|w \cdot \Delta^{\frac{n}{4}} w\|_{L^2(B_{16r})} + \|w^i \alpha_{ij} \Delta^{\frac{n}{4}} w^j\|_{L^2(B_{16r})} \\ \leq\ & \delta\left(\|\Delta^{\frac{n}{4}} w\|_{L^2(B_{\Lambda r})} + [w]_{B_{\Lambda r},\frac{n}{2}}\right) + C_{\delta,w}\left(r^{\frac{n}{2}} + \sum_{k=-\infty}^{\infty} 2^{-\gamma|k|}\left(\|\Delta^{\frac{n}{4}} w\|_{L^2(A_k)} + [w]_{A_k,\frac{n}{2}}\right)\right). \end{aligned}$$

In particular, as $|w| = 1$ on $B_{16r}(x_0) \subset \tilde{D}$ we have

$$\|\Delta^{\frac{n}{4}} w\|_{L^2(B_{16r})} \leq \delta\left(\|\Delta^{\frac{n}{4}} w\|_{L^2(B_{\Lambda r})} + [w]_{B_{\Lambda r},\frac{n}{2}}\right) + C_{\delta,w}\left(r^{\frac{n}{2}} + \sum_{k=-\infty}^{\infty} 2^{-\gamma|k|}\left(\|\Delta^{\frac{n}{4}} w\|_{L^2(A_k)} + [w]_{A_k,\frac{n}{2}}\right)\right). \tag{9.3}$$

Then, by Lemma 8.1 we have for a certain $\gamma > 0$ (possibly smaller than the one chosen before)

$$\begin{aligned} & [w]_{B_r,\frac{n}{2}} + \|\Delta^{\frac{n}{4}} w\|_{L^2(B_r)} \\ \overset{L.8.1}{\leq}\ & \varepsilon [w]_{B_{16r}} + C_\varepsilon \left(\|\Delta^{\frac{n}{4}} w\|_{L^2(B_{16r})} + \sum_{k=-\infty}^{\infty} 2^{-\gamma|k|}\left([w]_{A_k,\frac{n}{2}} + \|\Delta^{\frac{n}{4}} w\|_{L^2(A_k)}\right)\right) \\ \overset{(9.3)}{\prec}\ & \varepsilon [w]_{B_{16r}} + \delta C_\varepsilon \left(\|\Delta^{\frac{n}{4}} w\|_{L^2(B_{\Lambda r})} + [w]_{B_{\Lambda r},\frac{n}{2}}\right) + C_{\varepsilon,\delta,w}\left(r^{\frac{n}{2}} + \sum_{k=-\infty}^{\infty} 2^{-\gamma|k|}\left([w]_{A_k,\frac{n}{2}} + \|\Delta^{\frac{n}{4}} w\|_{L^2(A_k)}\right)\right). \end{aligned}$$

Thus, if we set $\delta := (C_\varepsilon)^{-1}\varepsilon$, the claim is proven.

Lemma 9.4 □

Finally, we can prove Theorem 1.2, which is an immediate consequence of the following theorem and the Euler-Lagrange-Equations, Lemma 7.1.

Theorem 9.5. *Let $w \in H^{\frac{n}{2}}(\mathbb{R}^n) \cap L^\infty$ as in Lemma 9.4. Then for any $E \subset \tilde{D}$ with positive distance from ∂D there is $\beta > 0$ such that $w \in C^{0,\beta}(E)$.*

Proof of Theorem 9.5.
Squaring the estimate of Lemma 9.4, we have for arbitrary $\varepsilon > 0$ some $\Lambda > 0$, $R > 0$ and $\gamma > 0$ and any $B_r(x) \subset \mathbb{R}^n$ where $B_{\Lambda r}(x) \subset \tilde{D}$, $r \in (0,R]$

$$\begin{aligned} \left([w]_{B_r,\frac{n}{2}}\right)^2 + \left(\|\Delta^{\frac{n}{4}} w\|_{L^2(B_r)}\right)^2 \ \leq\ & 4\varepsilon^2\left([w]^2_{B_{\Lambda r},\frac{n}{2}} + \|\Delta^{\frac{n}{4}} w\|^2_{L^2(B_{\Lambda r})}\right) \\ & + C_\varepsilon \left(\sum_{k=-\infty}^{\infty} 2^{-\gamma|k|}\left([w]^2_{A_k,\frac{n}{2}} + \|\Delta^{\frac{n}{4}} w\|^2_{L^2(A_k)}\right) + C_\varepsilon r^n \right), \end{aligned}$$

where $A_k \equiv A_k(r,x_0) = B_{2^{k+1}r}(x_0)\backslash B_{2^{k-1}r}(x_0)$. Set $a_k \equiv a_k(r,x) := [w]^2_{A_k,\frac{n}{2}} + \|\Delta^{\frac{n}{4}} w\|^2_{L^2(A_k)}$. Then, for some uniform $C_1 > 0$ and $c_1 > 0$ and some $K = K_\Lambda \in \mathbb{N}$

$$\|\Delta^{\frac{n}{4}} w\|^2_{L^2(B_{\Lambda r})} \leq C_1 \sum_{k=-\infty}^{K_\Lambda} a_k, \quad \text{and} \quad [w]^2_{B_{\Lambda r},\frac{n}{2}} \overset{L.8.2}{\leq} C_1 \sum_{k=-\infty}^{K_\Lambda} a_k,$$

and of course, $[w]^2_{B_r,\frac{n}{2}} + \|\Delta^{\frac{n}{4}} w\|^2_{L^2(B_r)} \geq c_1 \sum_{k=-\infty}^{-1} a_k$, as well as $\|a_k\|_{l^1(\mathbb{Z})} \prec \|\Delta^{\frac{n}{4}} w\|^2_{L^2(\mathbb{R}^n)}$. Choosing $\varepsilon > 0$ sufficiently small to absorb the effects of the independent constants c_1 and C_1, this implies

$$\sum_{k=-\infty}^{-1} a_k \leq \frac{1}{2} \sum_{k=-\infty}^{K_\Lambda} a_k + C \sum_{k=-\infty}^{\infty} 2^{-\gamma|k|} a_k + Cr^n \tag{9.4}$$

This is valid for any $B_r(x) \subset B_{\Lambda r}(x) \subset \tilde{D}$, where $r \in (0,R)$. Let E be a bounded subset of $\tilde{D}$ with proper distance to the boundary ∂D. Let $R_0 \in (0,R)$ such that for any $x \in E$ the ball $B_{2\Lambda R_0}(x) \subset \tilde{D}$. Fix some arbitrary $x \in E$. Let now for $k \in \mathbb{Z}$,

$$b_k \equiv b_k(x) := [w]^2_{A_k(\frac{R_0}{2}),\frac{n}{2}} + \|\Delta^{\frac{n}{4}} w\|^2_{L^2(A_k(\frac{R_0}{2}))} = a_k(\frac{R_0}{2}).$$

Then for any $N \leq 0$,

$$\sum_{k=-\infty}^{N} b_k \overset{(9.4)}{\leq} \frac{1}{2} \sum_{k=-\infty}^{K_\Lambda+N+1} b_k + C\ 2^\gamma \sum_{k=-\infty}^{\infty} 2^{-\gamma|k-N|} b_k + C\ R_0^n\ 2^{nN}$$

Consequently, by Lemma A.1, for a $N_0 < 0$ and a $\beta > 0$ (not depending on x),

$$\sum_{k=-\infty}^{N} b_k \leq C\ 2^{\beta N}, \quad \text{for any } N \leq N_0.$$

This implies in particular for $\tilde{R}_0 = 2^{N_0} R_0$ (again using Lemma 8.2)

$$[v]_{B_r(x_0),\frac{n}{2}} \leq C_{R_0}\ r^{\frac{\beta}{2}} \quad \text{for all } r < \tilde{R}_0 \text{ and } x \in E.$$

Finally, Dirichlet Growth Theorem, Theorem A.2, implies that $v \in C^{0,\beta}(E)$.

Theorem 9.5 □

A Appendix: Ingredients for the Dirichlet Growth Theorem

As a consequence of [DLR09, Proposition A.1] one checks that the following Iteration Lemma holds, too. For a proof we refer to [Sch10b, Appendix].

Lemma A.1. *For any $\Lambda_1, \Lambda_2, \gamma > 0$, $L \in \mathbb{N}$ there exists a constant $\Lambda_3 > 0$ and an integer $\bar{N} \leq 0$ such that the following holds. Let $(a_k) \in l^1(\mathbb{Z})$, $a_k \geq 0$ for any $k \in \mathbb{Z}$ such that for every $N \leq 0$,*

$$\sum_{k=-\infty}^{N} a_k \leq \frac{1}{2} \sum_{k=-\infty}^{N+L} a_k + \Lambda_1 \sum_{k=-\infty}^{N} 2^{\gamma(k-N)} a_k + \Lambda_2 \sum_{k=N+1}^{\infty} 2^{\gamma(N-k)} a_k + \Lambda_2 2^{\gamma N}.$$

Then for some $\beta \in (0,1)$, $\Lambda_4 > 0$ (depending only on $\|a_k\|_{l^1(\mathbb{Z})}$, Λ_3) and for any $N \leq \bar{N}$

$$\sum_{k=-\infty}^{N} a_k \leq \Lambda_4 2^{\beta N}.$$

Next, we will state a Dirichlet Growth-Type theorem whose proof uses mainly Poincaré's inequality. For more details we refer to [Sch10b, Appendix]. For an approach by potential analysis, we refer to [Ada75], in particular [Ada75, Corollary after Proposition 3.4].

Lemma A.2 (Dirichlet Growth Theorem)**.** *Let $D \subset \mathbb{R}^n$ be a smoothly bounded, convex domain, let $v \in H^{\frac{n}{2}}(\mathbb{R}^n)$ and assume there are constants $\Lambda > 0$, $\alpha \in (0,1)$, $R > 0$ such that*

$$\sup_{\substack{r \in (0,R) \\ x \in D}} r^{-\alpha} [v]_{B_r(x),\frac{n}{2}} < \Lambda.$$

Then $v \in C^{0,\alpha}(D)$.

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